Configuration spaces of an embedding torus and cubical spaces

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Abstract

For a smooth manifold M obtained as an embedding torus, $A \bigcup C \times [-1,1]$, we consider the ordered configuration space $\mathbb{F}_k(M)$ of k distinct points in M. We show that there is a homotopical cubical resolution of $\mathbb{F}_k(M)$ defined from the configuration spaces of A and C. From it, we deduce a universal method for the computation of the pure braid groups of a manifold. We illustrate the method in the case of the Möbius band.

Introduction

For a manifold M, the configuration space of k points in M is the space

$$\mathbb{F}_k(M) = \left\{(x_1, \dots, x_k) \in M^{\times k} \mid i \neq j \Rightarrow x_i \neq x_j\right\}.$$

As pointed-out by R. Fox and L. Neuwirth [4], configuration spaces are closely related to braid groups. In fact the fundamental group of the configuration spaces of the euclidean plane $\mathbb{R}^2$ is Artin's pure braid groups [1]. Hence, the pure braid group on k strands of a manifold M is naturally defined as the fundamental group of $\mathbb{F}_k(M)$.

The aim of this paper is two-folds. On one hand, we give an explicit homotopy model for the configuration spaces of a smooth manifold M obtained as an embedding torus described below, $A \bigcup C \times [-1,1]$. The model is given in terms of a homotopy cubical space, or $h\square$-space. On the other hand, using the previous model, we give a method for the determination of the braid groups of a manifold. To carry on such a calculation, one does not need to understand the full machinery behind the homotopy model. In fact, only basic knowledge on the fundamental group is required.

More precisely, let A and C be two manifolds of dimension n and $n-1$ respectively. Denote by ∂A the boundary of A and fix two disjoint embeddings $i^- : C \to \partial A$ and $i^+ : C \to \partial A$. The homotopy colimit of the diagram $C \underset{i^+}{\overset{i^-}{\rightrightarrows}} A$ is called an *embedding torus* and is a topological n-manifold with corners:

$$M = A \bigcup_{i^- \times \{-1\} \bigsqcup i^+ \times \{+1\}} C \times [-1,1] = \operatorname{hocolim}\left(C \underset{i^+}{\overset{i^-}{\rightrightarrows}} A \right).$$

In the sequel, the maps i^- and i^+ will be denoted by i^ϵ, respectively with $\epsilon = -1$ and $\epsilon = +1$. This situation is sufficiently general as the next example shows.

0.1 Examples

1. Let C be the interval $[-1,1]$ and A be the square $[-1,1]\times[-1,1]$.
   - If the maps i^ϵ are defined by $i^\epsilon(x)=(x,\epsilon)$ then M is a cylinder $\mathbb{R}^2\setminus\{0\}$.
   - If the maps i^ϵ are defined by $i^\epsilon(x)=(\epsilon x,\epsilon)$ then M is the Möbius band, denoted by $\mathcal{M}$.
2. The surfaces of genus $g\geq 1$, oriented and non-oriented, can be obtained with A the disk D^2 with $2g$ small disks removed and C the disjoint union of g copies of the circle S^1.
3. If U and V are two n-manifolds, the connected sum $U\#V$ can be obtained with $A=\left(U\setminus\mathring{D}^n\right)\bigsqcup\left(V\setminus\mathring{D}^n\right)$ and $C=S^{n-1}$.
4. If M is obtained from a manifold N by a surgery, then M can be obtained as an embedding torus with $A=(N\setminus S^r\times\mathring{D}^{n-r})\bigsqcup(D^{r+1}\times S^{n-r-1})$ and $C=S^r\times S^{n-r-1}$.

Main Theorem
There exists a homotopical cubical space $X_\bullet^k$, defined using the configuration spaces of A and C and the embeddings i^- and i^+, such that the geometrical realization $|X_\bullet^k|$ has the same homotopy type as $\mathbb{F}_k(M)$.

0.2 Example
When $k=2$, the homotopical cubical space model of $\mathbb{F}_2(M)$ is given by the diagram

$$X_\bullet^2:\ \mathbb{F}_2(C)\ \begin{array}{c}\xrightarrow{d_1^-}\\ \xrightarrow{d_1^+}\\ \xrightarrow{d_2^-}\\ \xrightarrow{d_2^+}\end{array}\ C\times A\bigsqcup A\times C\ \underset{d_1^+}{\overset{d_1^-}{\rightrightarrows}}\ \mathbb{F}_2(A)\ \text{ with } d_1^\epsilon\circ d_2^{\epsilon'}\simeq d_1^{\epsilon'}\circ d_1^\epsilon.$$

The paper is organized as follows. Section 1 introduces the notion of homotopical cubical space. The cubical resolution associated to the embedding torus M is described in Section 2. Finally, in Section 3, we explain the method of computation of the braid groups of M and detail the case of the Möbius band.

All spaces considered here are assumed to be compactly generated and with the homotopy type of a CW-complex. We denote by Top the category of such spaces.

Note: The section on braid groups can be read rather independently, only Example 1.10 and SubSection 2.1 are required to understand it.

1 Cubical Spaces

In this section, we define the notion of homotopical cubical spaces, or $h\Box$-spaces. Those spaces are defined as homotopy $\Box$-diagrams for a well chosen category $\Box$.

1.1 Homotopy limits and colimits

We first expose some results of R. Vogt [12, 11] concerning homotopy colimit of up to homotopy commutative diagram of spaces.

Let $\mathcal{C}$ be a small category.

1.1 Definition
For each $A \in \mathcal{C}$ and $B \in \mathcal{C}$, let

$$\mathcal{C}_n(A,B) = \{(f_n, f_{n-1}, \dots, f_1) \in (\mathrm{Mor}\,\mathcal{C})^n \mid f_n \circ \cdots \circ f_1 : A \to B \text{ in } \mathcal{C}\}, \quad n \in \mathbb{N}^*,$$
$$\mathcal{C}_0(A,B) = \begin{cases} \{id_A\} & A = B \\ \emptyset & A \neq B. \end{cases}$$

A *homotopy $\mathcal{C}$-diagram D*, or *$h\mathcal{C}$-diagram*, consists of

- a map $D_0 : \mathrm{Ob}\,\mathcal{C} \to \mathrm{Top}$,
- a collection of maps indexed by $\mathrm{Ob}\,\mathcal{C}$

  $$D_B : \coprod_{n \geq 0} \coprod_{A \in \mathcal{C}} \mathcal{C}_{n+1}(A,B) \times I^n \times D_0 A \to D_0 B$$

  where $I = [0,1]$ and such that:

  1. $D_B(id_B; x) = x$, for $x \in D_0 B$.
  2. For $n > 0$, $(f_n, f_{n-1}, \dots, f_0) \in \mathcal{C}_{n+1}(A,B)$ and $(t_n, t_{n-1}, \dots, t_1) \in I^n$,

  $$D_B(f_n, t_n, f_{n-1}, t_{n-1}, \dots, f_0; x) = \begin{cases} D_B(f_n, t_n, \dots, f_1; x) & f_0 = id, \\ D_B(f_n, t_n, \dots, f_{i+1}, t_{i+1}t_i, f_{i-1}, \dots, f_0; x) & f_i = id, \quad 0 < i < n, \\ D_B(f_{n-1}, t_{n-1}, \dots, f_0; x) & f_n = id, \\ D_B(f_n, t_n, \dots, f_{i+1}, t_{i+1}, f_i \circ f_{i-1}, t_{i-1}, \dots, f_0; x) & t_i = 1, \\ D_B(f_n, t_n, \dots, f_i; D_C(f_{i-1}, t_{i-1}, \dots, f_0; x)) & t_i = 0, \end{cases}$$

  with C the source of f_i.

When no confusion is possible, we write D in place of D_B. Roughly, an $h\mathcal{C}$-diagram is a kind of functor from $\mathcal{C}$ to Top in which the equality between $D(f \circ g)$ and $D(f) \circ D(g)$ is valid only up to a coherent homotopy, see [11, Example page 18]. The definition of the homotopy colimit of such a diagram states as follows.

1.2 Definition
Let D be a $h\mathcal{C}$-diagram. The *homotopy colimit* of D, denoted $\mathrm{hocolim}\, D$, is defined as the space

$$\mathrm{hocolim}\, D = \coprod_{A,B \in \mathcal{C}} \coprod_{n \geq 0} \mathcal{C}_n(A,B) \times I^n \times D_0 A \Big/ \sim$$

where the relation $\sim$ is given by

$$(t_n, f_n, \dots, t_1, f_1; x) = \begin{cases} (t_n, f_n, \dots, t_2, f_2; x) & f_1 = id, \\ (t_n, f_n, \dots, f_{i+1}, t_i t_{i-1}, f_{i-1}, \dots, f_1; x) & f_i = id, \quad 1 < i, \\ (t_n, f_n, \dots, t_{i+1}, f_{i+1} \circ f_i, t_{i-1}, \dots, f_1; x) & t_i = 1, \quad i < n, \\ (t_{n-1}, f_{n-1}, \dots, f_1; x) & t_n = 1, \\ (t_n, f_n, \dots, f_{i+1}; D_C(f_i, t_{i-1}, \dots, f_1; x)) & t_i = 0, \end{cases}$$

with C the source of f_i.

For $p \in \mathbb{N}$, the image of $\coprod_{A,B \in \mathcal{C}} \coprod_{n=0}^{p} \mathcal{C}_n(A,B) \times I^n \times D_0 A$ in $\mathrm{hocolim}\, D$ is denoted by $\mathrm{hocolim}^p\, D$.

1.2 The $\Box$ category

In this paragraph, we define the category $\Box$. We first introduce the sets $\mathcal{C}_p^n$, used in the definition of the morphisms of $\Box$.

1.3 Definition
Let $\lceil 1, n \rfloor$ be the set of integers between 1 and n and

$$\mathcal{C}_p^n = \{P \subset \lceil 1, n \rfloor \mid |P| = p\} = \{(i_1, \ldots, i_p) \in \lceil 1, n \rfloor^p \mid i_1 < \cdots < i_p\}$$

be the set of subsets of $\lceil 1, n \rfloor$ of cardinal p. We define the map $\complement : \mathcal{C}_p^n \to \mathcal{C}_{n-p}^n$ that sends an element $i = (i_1, \ldots, i_p)$ to its complement in $\lceil 1, n \rfloor$. We also define two associative operations $\wedge$ and $\vee$:
For $p < q < n$, $\wedge :$

$$\begin{array}{rcl} \mathcal{C}_q^p \times \mathcal{C}_p^n & \to & \mathcal{C}_q^n \\ (j, i) = ((j_1, \ldots, j_q), (i_1, \ldots, i_p)) & \mapsto & j \wedge i = (i_{j_1}, \ldots, i_{j_q}) \end{array}$$

and for $1 \leq p + q \leq k$, $\vee :$ $\begin{array}{rcl} \mathcal{C}_q^{n-p} \times \mathcal{C}_p^n & \to & \mathcal{C}_{p+q}^n \\ (j, i) & \mapsto & j \vee i = i + (j \wedge \complement i) \end{array}$
where $+$ stands for the union of sets in $\lceil 1, n \rfloor$. At last, we define a map $[\ ,\ ]$ acting like a partial inverse for the $\vee$ operation:

$$[\ ,\ ] : \begin{array}{rcl} \mathcal{C}_q^{n-p} \times \mathcal{C}_p^n & \to & \mathcal{C}_p^{p+q} \\ (j, i) & \mapsto & [j, i] \end{array}$$

where $[j, i]$ is the only element such that $[j, i] \wedge (j \vee i) = i$.

1.4 Examples

- If $i = (2, \mathbf{3}, 5, \mathbf{7}, 9, \mathbf{11}, 13, 17) \in \mathcal{C}_8^{18}$ and $j = (2, 4, 6) \in \mathcal{C}_3^8$, then
$$j \wedge i = (i_2, i_4, i_6) = (3, 7, 11) \in \mathcal{C}_3^{18}.$$
- If $i = (2, 3, 5, 7, 9, 11, 13, 17) \in \mathcal{C}_8^{18}$ and $j = (1, 4, 6, 9) \in \mathcal{C}_4^{10}$, then
$$\complement i = (1, 4, 6, 8, 10, 12, 14, 15, 16, 18) \text{ thus } j \wedge \complement i = (1, 8, 12, 16).$$
The set $j \vee i$ is given by
$$\begin{array}{rcl} i + (j \wedge \complement i) & = & (2, 3, 5, 7, 9, 11, 13, 17) + (1, 8, 12, 16) \\ & = & (1, 2, 3, 5, 7, 8, 9, 11, 12, 13, 16, 17) \in \mathcal{C}_{12}^{18}. \end{array}$$
In order to precise the set $[j, i] \in \mathcal{C}_8^{12}$, we have to determine the position of the integers forming $i = (2, 3, 5, 7, 9, 11, 13, 17)$ in $j \vee i = (1, \mathbf{2}, \mathbf{3}, \mathbf{5}, \mathbf{7}, 8, \mathbf{9}, \mathbf{11}, 12, \mathbf{13}, 16, \mathbf{17})$. We deduce $[j, i] = (2, 3, 4, 5, 7, 8, 10, 12)$.

We can use the operation $[\ ,\ ]$ to extend the definition of $\vee$ to a coloring of the integers forming i and j.

1.5 Definition
For $i = (i_1, \ldots, i_p) \in \mathcal{C}_p^n$, $\epsilon = (\epsilon_1, \ldots, \epsilon_p) \in \{-1, 1\}^p$, $j = (j_1, \ldots, j_q) \in \mathcal{C}_q^{n-p}$ and $\epsilon' = (\epsilon'_1, \ldots, \epsilon'_q) \in \{-1, 1\}^q$, we define $\epsilon'' = (\epsilon''_1, \ldots, \epsilon''_{p+q}) \in \{-1, 1\}^{p+q}$ as

$$\begin{array}{lll} \epsilon''_{l_\alpha} = \epsilon_\alpha & \text{with } l = [j, i] \in \mathcal{C}_p^{p+q} & \text{and } \alpha \in \lceil 1, p \rfloor, \\ \epsilon''_{s_\beta} = \epsilon'_\beta & \text{with } s = \complement l \in \mathcal{C}_q^{p+q} & \text{and } \beta \in \lceil 1, q \rfloor. \end{array}$$

When no confusion is possible, the element $\epsilon'' \in \{-1, 1\}^{p+q}$ is denoted by $\epsilon_{j \vee i}$.

1.6 Definition
The category $\square$ is defined by

$$\begin{array}{l} \mathrm{Ob}\,\square = \mathbb{N} \\ \mathrm{Mor}_{\square}(p,r) = \begin{cases} \left\{ f_i^{\epsilon_i} \mid i \in \mathcal{C}_p^{p-r}, \epsilon_i \in \{-1,1\}^{p-r} \right\} & p > r, \text{ with } f_j^{\epsilon_j} \circ f_i^{\epsilon_i} = f_{j\vee i}^{\epsilon_{j\vee i}}, \\ \{id_p\} & p = r, \\ \emptyset & p < r. \end{cases} \end{array}$$

For $p \in \mathbb{N}$, we also define $\square_p$ as the full subcategory of $\square$ whose objects are $\lceil 0, p \rfloor$.

We finally prove some assumptions on the operations defined on the spaces $\mathcal{C}_*^*$. These claims are convenient for proving the existence of a $h\square$-space structure.

1.7 Proposition

(1) For $j \in \mathcal{C}_q^p$ and $i \in \mathcal{C}_p^n$, $\quad \complement(j \wedge i) = (\complement j) \vee (\complement i)$.

(2) For $j \in \mathcal{C}_q^{n-p}$ and $i \in \mathcal{C}_p^n$, $\quad (\complement j) \wedge (\complement i) = \complement(j \vee i)$.

(3) For $j \in \mathcal{C}_q^{n-p}$ and $i \in \mathcal{C}_p^n$, $[j,i]$ is the only element of $\mathcal{C}_p^{p+q}$ which satisfies the relation

$$\complement[j,i] \wedge (j \vee i) = j \wedge \complement i.$$

(4) For $k \in \mathcal{C}_r^{n-p-q}$, $j \in \mathcal{C}_q^{n-p}$ and $i \in \mathcal{C}_p^n$,

$$[k \vee j, i] = [j,i] \wedge [k, j \vee i] \quad \textit{and} \quad [k, j \vee i] = [k,j] \vee [k \vee j, i].$$

Proof. Let $j \in \mathcal{C}_q^p$ and $i \in \mathcal{C}_p^n$, the relations (1) and (2) follow from the equalities

$$j \wedge i = (j \wedge i) \cap i = \complement(\complement j \wedge i) \cap i = \complement(\complement j \wedge i + \complement i) = \complement(\complement j \vee \complement i).$$

Let $k \in \mathcal{C}_r^{n-p-q}$, $j \in \mathcal{C}_q^{n-p}$ and $i \in \mathcal{C}_p^n$. Observe that $j \wedge \complement i = (j \vee i) \setminus i$ and $\complement k \wedge (j \vee i) = (j \vee i) \setminus (k \wedge (j \vee i))$. Hence, we have the equivalence

$$k \wedge (j \vee i) = i \Longleftrightarrow \complement k \wedge (j \vee i) = (j \vee i) \setminus i = j \wedge \complement i.$$

This shows the relation (3).

Let $k \in \mathcal{C}_r^{n-p-q}$, $j \in \mathcal{C}_q^{n-p}$ and $i \in \mathcal{C}_p^n$, we have

$$[j,i] \wedge [k, j \vee i] \wedge (k \vee j \vee i) = [j,i] \wedge (j \vee i) = i.$$

The first equality of (4) follows from the unicity of the element $[k \vee j, i]$. Furthermore, we have

$$\complement[k,j] \wedge \complement[k \vee j, i] \wedge (k \vee j \vee i) = \complement[k,j] \wedge (k \vee j) \wedge \complement i = k \wedge \complement j \wedge \complement i = k \wedge \complement(j \vee i).$$

Thus, the second equality of (4) follows from (3). $\square$

1.3 $h\square$-spaces

1.8 Definition
A *homotopical cubical space*, or *$h\square$-space*, is defined as a $h\square$-diagram. If $X_\bullet$ is a $h\square$-space, we depict this diagram as a cubical complex

$$\underset{d_5^+}{\overset{d_1^-}{\rightrightarrows}} X_4 \underset{d_4^+}{\overset{d_1^-}{\rightrightarrows}} X_3 \underset{d_3^+}{\overset{d_1^-}{\rightrightarrows}} X_2 \underset{d_2^+}{\overset{d_1^-}{\rightrightarrows}} X_1 \underset{d_1^+}{\overset{d_1^-}{\rightrightarrows}} X_0$$

with the conventions $X_i = X(i)$ and $d_i^\epsilon = X(f_i^\epsilon) : X_p \to X_{p-1}$ for $f_i^\epsilon \in \mathrm{Mor}(p, p-1)$. Finally *the geometrical realization* of $X_\bullet$, is defined as $\mathrm{hocolim}\, X_\bullet$ and denoted by $|X_\bullet|$. We define analogously a $h\square_p$-space as a $h\square_p$-diagram. We also denote by $X_{\leq p}$ the image of $\square_p$ by X. Observe that the new diagram $X_{\leq p}$ inherits a $h\square_p$-space structure from the $h\square$-space $X_\bullet$.

The geometrical realization of $X_\bullet$ can be obtained by induction in the following way (see Segal [8] for the simplicial case).

1.9 Proposition (Inductive construction of the geometrical realization)
Let $X_\bullet$ be $h\square$-space. For every $i \in \mathbb{N}^*$, there exists a map $\Phi_i : \partial([-1,1]^i) \times X_i \to |X_{\leq i}|$ such that the square

$$\begin{tikzcd}
\partial([-1,1]^i) \times X_i \arrow[r, "\Phi_i"] \arrow[d, hook] & {|X_{\leq i-1}|} \arrow[d, hook] \\
{[-1,1]^i \times X_i} \arrow[r] & {|X_{\leq i}|}
\end{tikzcd}$$

is a pushout. Furthermore, the geometrical realization of $X_\bullet$ is the colimit of

$$|X_{\leq 1}| \to |X_{\leq 2}| \to \cdots \to |X_{\leq i}| \to |X_{\leq i+1}| \to \cdots$$

1.10 Example (Geometrical realization of a $h\square_2$-space)
Start with a $h\square_2$-space

$$X_\bullet :\ X_2 \overset{d_1^-,\, d_1^+,\, d_2^-,\, d_2^+}{\rightrightarrows\rightrightarrows} X_1 \underset{d_1^+}{\overset{d_1^-}{\rightrightarrows}} X_0\,.$$

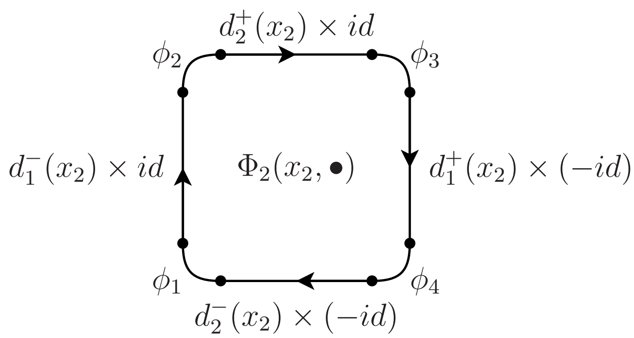
From the definition of homotopy colimit (1.2), we have $|X_{\leq 0}| = X_0$ and

$$\begin{aligned}
|X_{\leq 1}| &= \mathrm{hocolim}\left(X_1 \underset{d_1^+}{\overset{d_1^-}{\rightrightarrows}} X_0 \right) \\
&= X_1 \times [-1,1] \bigcup_{(x_1,\epsilon)\sim d_1^\epsilon(x_1)} X_0 \\
&= \mathrm{colim}\,(X_1 \leftarrow X_1 \times \{-1,1\} \to |X_{\leq 0}|)\,.
\end{aligned}$$

The $h\square_2$-space $X_\bullet$ is the image of the category $\square_2$ represented below

$$\begin{tikzcd}
0 & 1 \arrow[l, "f_1^-"'] \arrow[r, "f_1^+"] & 0 \\
1 \arrow[u, "f_1^+"] \arrow[d, "f_1^-"'] & 2 \arrow[ul, "f_{1,2}^{-+}"] \arrow[u, "f_2^+"] \arrow[ur, "f_{1,2}^{++}"] \arrow[l, "f_1^-"] \arrow[r, "f_1^+"] \arrow[dl, "f_{1,2}^{--}"] \arrow[d, "f_2^-"] \arrow[dr, "f_{1,2}^{+-}"] & 1 \arrow[u, "f_1^+"'] \arrow[d, "f_1^-"] \\
0 & 1 \arrow[l, "f_1^-"] \arrow[r, "f_1^+"] & 0
\end{tikzcd}$$

where all of the four external edges of the form $\square_1 : \; 0 \longleftarrow f_1^- \longrightarrow 1 \longrightarrow f_1^+ \longrightarrow 0$ are identified. Observe that the structure of $h\square_2$-space is equivalent to the existence of homotopies $d_1^- d_2^- \simeq d_1^- d_1^-$, $d_1^+ d_1^- \simeq d_1^- d_2^+$, $d_1^+ d_2^+ \simeq d_1^+ d_1^+$ and $d_1^- d_1^+ \simeq d_2^- d_1^+$. Hence for every $x_2 \in X_2$, we can define a path ϕ_1 of $|X_{\leq 1}|$ as the restriction at x_2 of the homotopy between $d_1^- d_2^-$ and $d_1^- d_1^-$. Similarly, define ϕ_2, ϕ_3 and ϕ_4 as the restrictions of the homotopies $d_1^+ d_1^- \simeq d_1^- d_2^+$, $d_1^+ d_2^+ \simeq d_1^+ d_1^+$ and $d_1^- d_1^+ \simeq d_2^- d_1^+$. Also denote by id the identity on $[-1,1]$. One construct a loop $\Phi_2(x_2, \bullet)$ in $|X_{\leq 1}|$ as follows:



More precisely, $\Phi_2(x_2, \bullet)$ is the following composition of paths:

$$\Phi_2(x_2, \bullet) = \phi_1 * (d_1^-(x_2) \times id) * \phi_2 * (d_2^+(x_2) \times id) * \phi_3 * (d_1^+(x_2) \times (-id)) * \phi_4 * (d_2^-(x_2) \times (-id)).$$

We extend it as a map $\Phi_2 : X_2 \times \partial[-1,1]^2 \to |X_{\leq 1}|$. Then we have a description of $|X_{\leq 2}|$ as a pushout:

$$\begin{tikzcd}
X_2 \times \partial[-1,1]^2 \arrow[r, "\Phi_2"] \arrow[d] & {|X_{\leq 1}|} \arrow[d] \\
X_2 \times [-1,1]^2 \arrow[r] & {|X_{\leq 2}|}
\end{tikzcd}$$

Proof (1.9). Consider a $h\square_i$-space $X_\bullet$. We describe the geometrical realization of $|X_{\leq i}|$ using the spaces $|X_{\leq i-1}|$ and X_i. Define the space $A_i = \displaystyle\coprod_{\substack{0 \leq j \leq i \\ 0 \leq n \leq i}} \mathcal{C}_n(i,j) \times I^n \times X_i \Big/ \sim$, where

the relations $\sim$ are given by

$$
\begin{aligned}
&(t_n, f_n, \ldots, t_1, f_1; x) \\
&\quad\sim \begin{cases}
(t_n, f_n, \ldots, t_2, f_2; x) & f_1 = id, \\
(t_n, f_n, \ldots, f_{i+1}, t_i t_{i-1}, f_{i-1}, \ldots, f_1; x) & f_i = id, \quad 1 < i, \\
(t_n, f_n, \ldots, t_{i+1}, f_{i+1} \circ f_i, t_{i-1}, \ldots, f_1; x) & t_i = 1, \quad\;\; i < n, \\
(t_{n-1}, f_{n-1}, \ldots, f_1; x) & t_n = 1.
\end{cases}
\end{aligned}
$$

Observe that in each relation, the element $x \in X_i$ remains unmodified. Since all the spaces are assumed to be in Top, this implies

$$
A_i = \coprod_{\substack{0 \le j \le i \\ 0 \le n \le i}} \mathcal{C}_n(i,j) \times I^n \times X_i \Big/ \sim \; = \left(\coprod_{\substack{0 \le j \le i \\ 0 \le n \le i}} \mathcal{C}_n(i,j) \times I^n \Big/ \sim \right) \times X_i = [-1,1]^i \times X_i .
$$

From Definition 1.2 of the homotopy colimit, we deduce that $|X_{\le i}|$ is obtained as the quotient space $A_i \coprod |X_{\le i-1}| \Big/ \sim$ where the relation $\sim$ is given by

$$
(t_n, f_n, \ldots, t_1, f_1; x) \sim (t_n, f_n, \ldots, f_{i+1}; D_C(f_i, t_{i-1}, \ldots, f_1; x))
$$

whenever $t_i = 0$ with C the source of f_{i+1}. We can rephrase this result by saying that there exists a map Φ_i such that the square below is a homotopy pushout.

$$
\begin{tikzcd}
\partial([-1,1]^i) \times X_i \arrow[r, "\Phi_i"] \arrow[d, hook] & {|X_{\le i-1}|} \arrow[d, hook] \\
{[-1,1]^i \times X_i} \arrow[r] & {|X_{\le i}|}
\end{tikzcd}
$$

□

As a direct consequence, we have the next result on the determination of the fundamental group of the geometrical realization of a $h\Box$-space.

1.11 Proposition
Let $X_\bullet$ be a $h\Box$-space, then the following spaces have the same fundamental group:

- $|X_\bullet|$,
- $|X_{\le 2}|$,
- *The geometrical realization of the square space* $X_2^{(0)} \mathrel{\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow \\ \longrightarrow}} X_1 \rightrightarrows X_0$.

with $X_2^{(0)}$ a 0-skeleton of the space X_2.

A last property on $h\Box$-spaces is required in order to prove the Main Theorem.

1.12 Proposition
Let X be a $h\Box$-space of the form

$$
\underset{d_5^+}{\overset{d_1^-}{\vdots}} \coprod_{i+j=4} X_{i,j} \underset{d_4^+}{\overset{d_1^-}{\vdots}} \coprod_{i+j=3} X_{i,j} \underset{d_3^+}{\overset{d_1^-}{\vdots}} \coprod_{i+j=2} X_{i,j} \underset{d_2^+}{\overset{d_1^-}{\vdots}} X_{0,1} \coprod X_{1,0} \underset{d_1^+}{\overset{d_1^-}{\rightrightarrows}} X_{0,0}
$$

such that each map $d_\alpha^\epsilon : X_{p,q} \to X_{p,q-1} \bigsqcup X_{p-1,q}$ *restricts to*

$$d_\alpha^\epsilon : X_{p,q} \to X_{p,q-1} \quad \textit{if} \quad \alpha \leq q \quad \textit{and} \quad d_\alpha^\epsilon : X_{p,q} \to X_{p-1,q} \quad \textit{if} \quad \alpha > q.$$

Then, we can form the following diagram

$$\begin{tikzcd}
(X') : \arrow[r, shift left, "d_1^-"] \arrow[r, shift right, "d_4^+"'] & X_{3,0} \arrow[r, shift left, "d_1^-"] \arrow[r, shift right, "d_3^+"'] & X_{2,0} \arrow[r, shift left, "d_1^-"] \arrow[r, shift right, "d_2^+"'] & X_{1,0} \arrow[r, shift left, "d_1^-"] \arrow[r, shift right, "d_1^+"'] & X_{0,0} \\
\arrow[r, shift left, "d_2^-"] \arrow[r, shift right, "d_5^+"'] & X_{3,1} \arrow[u, shift right, "d_1^+"] \arrow[u, shift left, "d_1^-"'] \arrow[r, shift left, "d_2^-"] \arrow[r, shift right, "d_4^+"'] & X_{2,1} \arrow[u, shift right, "d_1^+"] \arrow[u, shift left, "d_1^-"'] \arrow[r, shift left, "d_2^-"] \arrow[r, shift right, "d_3^+"'] & X_{1,1} \arrow[u, shift right, "d_1^+"] \arrow[u, shift left, "d_1^-"'] \arrow[r, shift left, "d_2^-"] \arrow[r, shift right, "d_2^+"'] & X_{0,1} \arrow[u, shift right, "d_1^+"] \arrow[u, shift left, "d_1^-"'] \\
\arrow[r, shift left, "d_3^-"] \arrow[r, shift right, "d_6^+"'] & X_{3,2} \arrow[u, shift right, "d_2^+"] \arrow[u, shift left, "d_1^-"'] \arrow[r, shift left, "d_3^-"] \arrow[r, shift right, "d_5^+"'] & X_{2,2} \arrow[u, shift right, "d_2^+"] \arrow[u, shift left, "d_1^-"'] \arrow[r, shift left, "d_3^-"] \arrow[r, shift right, "d_4^+"'] & X_{1,2} \arrow[u, shift right, "d_2^+"] \arrow[u, shift left, "d_1^-"'] \arrow[r, shift left, "d_3^-"] \arrow[r, shift right, "d_3^+"'] & X_{0,2} \arrow[u, shift right, "d_2^+"] \arrow[u, shift left, "d_1^-"'] \\
& {} \arrow[u, shift right, "d_3^+"] \arrow[u, shift left, "d_1^-"'] & {} \arrow[u, shift right, "d_3^+"] \arrow[u, shift left, "d_1^-"'] & {} \arrow[u, shift right, "d_3^+"] \arrow[u, shift left, "d_1^-"'] & {} \arrow[u, shift right, "d_3^+"] \arrow[u, shift left, "d_1^-"']
\end{tikzcd}$$

where each line and each column inherit a $h\square$*-space structure. Let* $|X_{j,\bullet}|$ *(resp.* $|X_{\bullet,j}|$*) be the geometrical realization of the cubical space on the* j*-th column (resp. row).*
Then, the geometrical realization of $X_\bullet$ *is the same as the one of any of the* $h\square$*-spaces*

$$\underset{d_4^+}{\overset{d_1^-}{\rightrightarrows}} |X_{3,\bullet}| \underset{d_3^+}{\overset{d_1^-}{\rightrightarrows}} |X_{2,\bullet}| \underset{d_2^+}{\overset{d_1^-}{\rightrightarrows}} |X_{1,\bullet}| \underset{d_1^+}{\overset{d_1^-}{\rightrightarrows}} |X_{0,\bullet}| \quad \textit{and} \quad \underset{d_4^+}{\overset{d_1^-}{\rightrightarrows}} |X_{\bullet,3}| \underset{d_3^+}{\overset{d_1^-}{\rightrightarrows}} |X_{\bullet,2}| \underset{d_2^+}{\overset{d_1^-}{\rightrightarrows}} |X_{\bullet,1}| \underset{d_1^+}{\overset{d_1^-}{\rightrightarrows}} |X_{\bullet,0}| \,.$$

Proof. The proof is a straightforward application of [12, theorem 2.4] once we have observed that the diagram (X') admits a structure of $h\square \times h\square$-diagram (see [12, Definition 2.1]) since it is defined from the $h\square$-diagram X. $\square$

2 Cubical resolution associated to an embedding torus

The aim of this section is to prove the Main Theorem. In Section 2.1, we give a geometric description of the resolution $X_\bullet^k$ whereas in Section 2.2 we prove that the $h\square$-space $X_\bullet^k$ is well defined and augmented. The section 2.3 is focused on the proof of the Main Theorem.

2.1 $h\square_2$-space associated to an embedding torus

Recall that

$$\mathcal{C}_p^k = \{P \subset \lceil 1, k \rfloor \mid |P| = p\} = \{(i_1, \dots, i_p) \in \lceil 1, k \rfloor^p \mid i_1 < \cdots < i_p\}\,.$$

We set $X_p^k = \mathcal{C}_p^k \times \mathbb{F}_p(C) \times \mathbb{F}_{k-p}(A)$. This space is naturally identified with the space of configurations of k points in $A \bigsqcup C$ with exactly p points in C. In this section, we define some maps $d_\alpha^\epsilon : X_p^k \to X_{p-1}^k$, $\alpha \in \lceil 1, p \rfloor$, $\epsilon \in \{-1, +1\}$ and give a geometric interpretation of them. They will form a first step toward the definition of the resolution $X_\bullet$ of $\mathbb{F}_k(M)$.

Recall that A is a manifold with boundaries. The collar neighborhood theorem [6, Theorem 17.1] asserts the existence of an embedding $f : \partial A \times [0, 2] \to A$ that extends the natural

inclusion $\partial A \times \{0\} \hookrightarrow A$. Later on, we identify a point $(a,t) \in \partial A \times [0,2]$ with its image by f. This embedding allows us to define a map r used in our construction. Geometrically, this map crushes down the collar neighborhood $\partial A \times [0,2]$ linearly onto $\partial A \times [1,2]$ leaving the complement unchanged.

2.1 Proposition
There is an injective map $r : A \to A \setminus (\partial A \times [0,1[)$ which is a weak deformation retract of the natural inclusion.

Proof. Define the map $R : A \times [0,2] \to A$ such that for $(a,s) \in A \times [0,2]$:

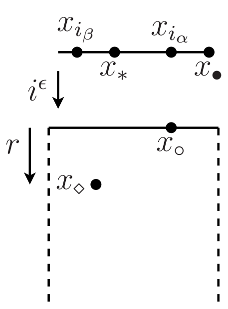
$$\begin{array}{lll} & R(a,s) = a & \text{if } a \in A \setminus (\partial A \times [0,2[), \\ \text{and} & R((a,t),s) = f(a, s + (2-s)t/2) & \text{if } (a,t) \in \partial A \times [0,2]. \end{array}$$

The map $r = R(\bullet, 1)$ fulfills the requirement. $\square$

2.2 Definition
For $1 \le \alpha \le p$, the map $d^\epsilon_\alpha : X^k_p \to X^k_{p-1}$ is defined by:

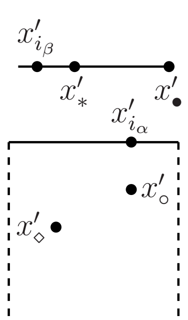
$$\begin{array}{c} \mathcal{C}^k_p \times \mathbb{F}_p(C) \times \mathbb{F}_{k-p}(A) \longrightarrow \mathcal{C}^k_{p-1} \times \mathbb{F}_{p-1}(C) \times \mathbb{F}_{k-p+1}(A) \\ ((i_1, \ldots, i_p), (c_1, \ldots, c_p), (a_1, \ldots, a_{k-p})) \\ \downarrow \\ \Big((i_1, .., \widehat{i_\alpha}, .., i_p), (c_1, .., \widehat{c_\alpha}, .., c_p), (r(a_1), .., r(a_{i_\alpha - \alpha}), i^\epsilon(c_\alpha), r(a_{i_\alpha - \alpha + 1}), .., r(a_{k-p})) \Big) \end{array}$$

We give now a geometric interpretation of the maps $d^\epsilon_\alpha : X^k_p \to X^k_{p-1}$.



Let $((i_1, \ldots, i_p), (c_1, \ldots, c_p), (a_1, \ldots, a_{k-p})) \in X^k_p$. As already stated, this object is identified with a configuration of k points, $(x_1, \ldots, x_k)$, living in $A \bigsqcup C$ with exactly p of them in C. On the picture on the left, the map i^ϵ is given by the natural inclusion of C (represented by the interval at the top of the picture) into ∂A (represented by the other non dashed interval). The retraction r sends down vertically the points living in A.

The action of the map d^ϵ_α on the above configuration consists of two successive steps:

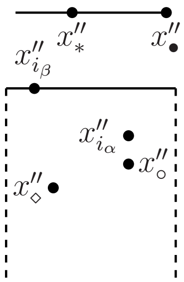


1. It pushes away from the boundaries all points in A using the retraction r,
2. It sends the α-th point living in C, $x_{i_\alpha} = c_\alpha$, into ∂A using the map i^ϵ.

Let $(x'_1, \ldots, x'_k)$ denotes the new configuration obtained, in particular, $x'_{i_\alpha} = i^\epsilon(x_{i_\alpha}) = i^\epsilon(c_\alpha)$. Observe that the use of the retraction is necessary since one cannot ensure that the point $i^\epsilon(c_\alpha)$ is not already in the configuration $(x_1, \ldots, x_k)$.

Now if we assume $\beta > \alpha$, then $c_\beta = x'_{i_\beta}$ is the $(\beta - 1)$-th point living in C in the configuration

$(x'_1, \ldots, x'_k)$.



Applying the map $d^\epsilon_{\beta-1}$, we get the result pictured on the left. The new configuration represents the action of $d^\epsilon_{\beta-1} \circ d^\epsilon_\alpha$ on $(x_1, \ldots, x_k)$ and is labeled $(x''_1, \ldots, x''_k)$ on the picture.

Starting from the configuration $(x_1, \ldots, x_k)$, we can also send x_{i_β} first in A and after send x_{i_α}. This leads to the action of the map $d^\epsilon_\alpha \circ d^\epsilon_\beta$ on the configuration $(x_1, \ldots, x_k)$. We can observe that $d^\epsilon_{\beta-1} \circ d^\epsilon_\alpha$ and $d^\epsilon_\alpha \circ d^\epsilon_\beta$ do not agree exactly but up to homotopy. The homotopy consists of sliding the points x''_{i_α} and x''_{i_β} along the retraction r as sketched below with dotted arrows.

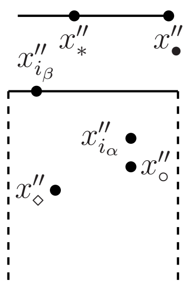
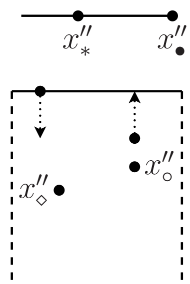


Image under $d^\epsilon_{\beta-1} \circ d^\epsilon_\alpha$



Sliding along the retraction

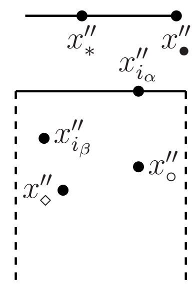


Image under $d^\epsilon_\alpha \circ d^\epsilon_\beta$

Observe that for both compositions, the point x''_{i_α} is of the form $(i^\epsilon(x_{i_\alpha}), t_\alpha) \in \partial A \times [0, 2[\subset A$ and similarly x''_{i_β} is of the form $(i^\epsilon(x_{i_\beta}), t_\beta)$. The preceding homotopy consists in modifying the t_* components of those points. It is well defined since it is assumed that $i^\epsilon(x_{i_\alpha}) \neq i^\epsilon(x_{i_\beta})$. As a consequence, we have a homotopy between the two configurations.

2.3 Proposition
If $\alpha < \beta$ and $(\epsilon', \epsilon) \in \{-1, +1\}^2$, then $d^\epsilon_\alpha \circ d^{\epsilon'}_\beta \simeq d^{\epsilon'}_{\beta-1} \circ d^\epsilon_\alpha$.

A more general statement is proved in next section. It provides the coherences between the preceding homotopies required to give a structure of $h\Box$-space to $X^k_\bullet$.

2.2 $h\Box$-space associated to an embedding torus

In this paragraph, we detail the $h\Box$-space structure of $X^k_\bullet$. More precisely, we exhibit maps satisfying the relations appearing in Definition 1.1.

2.4 Proposition
$X_\bullet$ admits a $h\Box$-space structure.

Proof. We now prove that $X_\bullet$ admits a $h\Box$-space structure as defined in Section 1. To prove this result, we introduce maps b^*_*. Let $i = (i_1, \ldots, i_q) \in \mathcal{C}^p_q$ and $\epsilon = (\epsilon_1, \ldots, \epsilon_q) \in \{-1, 1\}^q$, $q < p$. The map b^ϵ_i is defined as:

$$\begin{array}{llcl} b^\epsilon_i : & \mathcal{C}^k_p \times \mathbb{F}_p(C) \times \mathbb{F}_{k-p}(\mathring{A}) & \to & \mathcal{C}^k_{p-q} \times \mathbb{F}_{p-q}(C) \times \mathbb{F}_{k-p+q}(A) \\ & (m, (c_1, \ldots, c_p), (a_1, \ldots, a_{k-p})) & \mapsto & ((\complement i) \wedge m, (c_{n_1}, \ldots, c_{n_{p-q}}), (a'_1, \ldots, a'_{k-p+q})) \end{array}$$

$$\text{with } n=(n_1,\dots,n_{p-q})=\complement i\in\mathcal{C}^p_{p-q},\quad l=[i,\complement m]\in\mathcal{C}^{k-p+q}_{k-p},\quad s=\complement l\in\mathcal{C}^{k-p+q}_q,$$
$$a'_{l_\alpha}=a_\alpha\quad\text{and}\quad a'_{s_\beta}=i^{\epsilon_\beta}(c_{i_\beta}),\quad \alpha\in\lceil 1,k-p\rfloor,\beta\in\lceil 1,q\rfloor.$$

Observe that we cannot compose two maps b^ϵ_i ; for doing that, we need to use the retraction $r=R(\bullet,1)$ defined before (2.1). For instance, the map d^ε_α introduced in Definition 2.2 agrees with the composite $b^{(\varepsilon)}_{(\alpha)}\circ R(\bullet,1)$. In that case, $q=1$. For a general $q\geq 1$, the map $b^\epsilon_i\circ R(\bullet,1)$ has a geometrical description analogous to the one given in Section 2.1, but sends q points of C into A instead of just 1. We now detail the composite with the general retraction $R(\bullet,1-t)$.

Let $t\in]0,1]$, $j\in\mathcal{C}^{p-q}_r$ and $\epsilon'=(\epsilon'_1,\dots,\epsilon'_r)\in\{-1,1\}^r$. Consider the composition of maps $b^{\epsilon'}_j\circ R(\bullet,1-t)\circ b^\epsilon_i$, a straightforward calculus shows that this composition is described as follows:

$$\begin{array}{ccc}\mathcal{C}^k_p\times\mathbb{F}_p(C)\times\mathbb{F}_{k-p}(\mathring{A}) & \to & \mathcal{C}^k_{p-q-r}\times\mathbb{F}_{p-q-r}(C)\times\mathbb{F}_{k-p+q+r}(A)\\ (m,(c_1,\dots,c_p),(a_1,\dots,a_{k-p-r})) & \mapsto & ((\complement j)\wedge(\complement i)\wedge m,(c_{n''_1},\dots,c_{n''_{p-q-r}}),(a''_1,\dots,a''_{k-p+q+r}))\end{array}$$

where

- $n'=(n'_1,\dots,n'_{p-q-r})=\complement j\in\mathcal{C}^{p-q}_{p-q-r}$, $n''=(n_{n'_1},\dots,n_{n'_{p-q-r}})=n'\wedge n=\complement(j\vee i)\in\mathcal{C}^p_{p-q-r}$,
- $a''_{l'_\alpha}=R(a'_\alpha,1-t)$, with $l'=[j,i\vee\complement m]\in\mathcal{C}^{k-p+q+r}_{k-p+q}$ and $\alpha\in\lceil 1,k-p+q\rfloor$, which implies
$$a''_{l'_{l_\alpha}}=R(a_\alpha,1-t)\text{ if }\alpha\in\lceil 1,k-p\rfloor\text{ and }a''_{l'_{s_\alpha}}=R(i^{\epsilon_\alpha}(c_{i_\alpha}),1-t)\text{ if }\alpha\in\lceil 1,q\rfloor,$$
- $a''_{s'_\alpha}=i^{\epsilon'_\alpha}(c_{n_{j_\alpha}})$, with $s'=\complement l'\in\mathcal{C}^{k-p+q+r}_r$ and $\alpha\in\lceil 1,r\rfloor$.

Observe that the expression of the elements of the form a''_* shows that $b^{\epsilon'}_j\circ R(\bullet,1-t)\circ b^\epsilon_i$ can be extended to $t=1$ by a continuous map. In that case, we have:

- $a''_{l''_\alpha}=a_\alpha$, for $\alpha\in\lceil 1,k-p\rfloor$ and $l''=l\wedge l'=[i,\complement m]\wedge[j,i\vee\complement m]=[j\vee i,\complement m]\in\mathcal{C}^{k-p+q+r}_{k-p}$.
- $a''_{s''_\alpha}=i^{\epsilon''_\alpha}(c_{w_\alpha})$ with $\left|\begin{array}{l}s''=s'+s\wedge l'=s'+s\wedge\complement s'=s\vee s'=\complement(l\wedge l')=\complement(l''),\\ w=j\wedge n+i=j\wedge\complement i+i=j\vee i,\\ \epsilon''=\epsilon_{j\vee i}.\end{array}\right.$

Since $(\complement j)\wedge(\complement i)\wedge m=\complement(j\vee i)\wedge m$, we recognize exactly the map $b^{\epsilon_{j\vee i}}_{j\vee i}$. Finally, for each element

$$(f^{\epsilon_n}_{\underline{i}_n},t_n,f^{\epsilon_{n-1}}_{\underline{i}_{n-1}},t_{n-1},\dots,t_2,f^{\epsilon_1}_{\underline{i}_1};x)\in(\mathrm{Mor}_\square)_n(p,p-q)\times I^{n-1}\times(\mathcal{C}^k_p\times\mathbb{F}_p(C)\times\mathbb{F}_{k-p}(A))$$

define $X_{p-q}(f^{\epsilon_n}_{\underline{i}_n},t_n,f^{\epsilon_{n-1}}_{\underline{i}_{n-1}},t_{n-1},\dots,t_2,f^{\epsilon_1}_{\underline{i}_1};x)=$
$$b^{\epsilon_n}_{\underline{i}_n}\circ R(\bullet,1-t_n)\circ b^{\epsilon_{n-1}}_{\underline{i}_{n-1}}\circ R(\bullet,1-t_{n-1})\circ\cdots\circ R(\bullet,1-t_2)\circ b^{\epsilon_1}_{\underline{i}_1}\circ R(x,1).$$
The previous calculus of $b^{\epsilon'}_j\circ R(\bullet,1-t)\circ b^\epsilon_i$ shows that the maps X_* are well defined, give a coherent system of homotopy, and thus define a structure of $h\square$-space on $X_\bullet$. $\square$

Let ι denotes the inclusion of A into M. Since ι is an injective map, it extends to a map, labeled d_0, from $\mathbb{F}_k(A)$ into $\mathbb{F}_k(M)$.

2.5 Proposition
There exists a homotopy between $d_0 \circ d_1^+$ and $d_0 \circ d_1^-$ from $\mathcal{C}_1^k \times C \times \mathbb{F}_{k-1}(A)$ to $\mathbb{F}_k(M)$. Hence, the h$\square$-space $X_\bullet^k$ is augmented by d_0 and there is a map $\chi : |X_\bullet^k| \to \mathbb{F}_k(M)$ induced by this augmentation.

In the next section, we will prove the main Theorem by showing that the map χ is a homotopy equivalence.

Proof. Let g be the inclusion of $C \times [-1,1]$ into M. If $(i, c, (a_1, \ldots, a_{k-1})) \in \mathcal{C}_1^k \times C \times \mathbb{F}_{k-1}(A)$, then

$$d_0 \circ d_1^\epsilon(i, c, (a_1, \ldots, a_{k-1})) = d_0(r(a_1), \ldots, r(a_{i-1}), i^\epsilon(c), r(a_i), \ldots, r(a_{k-1})).$$

Define the map $H : \mathcal{C}_1^k \times C \times \mathbb{F}_{k-1}(A) \times [-1,1] \to \mathbb{F}_k(M)$ that sends $(i, c, (a_1, \ldots, a_{k-1}), t)$ to

$$(\iota \circ R(a_1, |t|), \ldots, \iota \circ R(a_{i-1}, |t|), g(c,t), \iota \circ R(a_i, |t|), \ldots, \iota \circ R(a_{k-1}, |t|))\,.$$

This is a homotopy between $H(\bullet, -1) = d_0 \circ d_1^-$ and $H(\bullet, +1) = d_0 \circ d_1^+$. Still, we have to check that H is well defined. For that, we observe:

- The elements $\iota \circ R(a_*, |t|)$ are all distinct since $\iota \circ R(\bullet, |t|)$ is injective by construction.
- The elements $\iota \circ R(a_*, |t|)$ are in A and the element $g(c,t)$ is in $C \times [-1,1]$. The intersection $C \times [-1,1] \cap A$ is always included in ∂A. But now, we have

  $$g(c,t) \in \partial A \Leftrightarrow t = \pm 1 \quad \text{and} \quad \iota(R(a_i), |t|) \in \partial A \Leftrightarrow t = 0.$$

  therefore, the point $g(c,t) \in C \times [-1,1]$ is always distinct of the points $\iota \circ R(a_*, |t|)$. $\square$

2.3 Proof of Main Theorem

The proof relies on the Fadell-Neuwirth fibration [2]: for a manifold A without boundaries, the projection $\pi_{k,p}$ of the first p components of $\mathbb{F}_k(A)$ onto $\mathbb{F}_p(A)$, $p < k$, is a fiber bundle with fiber over $(a_1, \ldots, a_p) \in \mathbb{F}_p(A)$ the space $\mathbb{F}_{k-p}(A \setminus \{a_1, \ldots, a_p\})$. Observe that this statement becomes false if A is a manifold with boundary. The following statement solves this problem by replacing the fiber by an homotopy fiber. It also justifies that the configuration spaces of a manifold and of its interior have the same homotopy type.

2.6 Proposition
Let A be a manifold of interior $\mathring{A}$. Denote by $r : A \to A \setminus (\partial A \times [0,1[)$ the retraction introduced in Proposition 2.1. The natural inclusions $\mathbb{F}_k(\mathring{A}) \overset{i}{\hookrightarrow} \mathbb{F}_k(A)$ are fiber homotopy equivalences. In particular, the homotopy fiber of $\mathbb{F}_k(A) \to \mathbb{F}_p(A)$ over an element $(a_1, \ldots, a_p) \in \mathbb{F}_p(A)$ is $\mathbb{F}_{k-p}(A \setminus \{r(a_1), \ldots, r(a_p)\})$.

Proof. Observe that the following square is commutative.

$$\begin{tikzcd} \mathbb{F}_k(\mathring{A}) \arrow[r, hook] \arrow[d] & \mathbb{F}_k(A) \arrow[d] \\ \mathbb{F}_p(\mathring{A}) \arrow[r, hook] & \mathbb{F}_p(A) \end{tikzcd}$$

Furthermore, the map $R : A \times [0,2] \to A$ defined in 2.1 shows that the composition of two successive maps in the line below is homotopic to the identity.

$$\begin{tikzcd}
\mathbb{F}_k(\mathring{A}) \arrow[r, hook, "i"] \arrow[rr, bend left, "R(\bullet,1)\simeq R(\bullet,0)=Id_{\mathbb{F}_k(\mathring{A})}"] & \mathbb{F}_k(A) \arrow[r, "R(\bullet,1)"] \arrow[rr, bend right, "R(\bullet,1)\simeq R(\bullet,0)=Id_{\mathbb{F}_k(A)}"'] & \mathbb{F}_k(\mathring{A}) \arrow[r, hook, "i"] & \mathbb{F}_k(A)
\end{tikzcd}$$

This proves that $\mathbb{F}_k(\mathring{A}) \hookrightarrow \mathbb{F}_k(A)$ is a fiber homotopy equivalence. $\square$

We prove now the Main Theorem and precise the maps between the geometrical realization $|X^k_\bullet|$ and $\mathbb{F}_k(M)$.

2.7 Theorem
The map χ, induced by the augmentation d_0, is a homotopy equivalence between the geometrical realization $\left|X^k_\bullet\right|$ and the configuration space $\mathbb{F}_k(M)$:

$$\operatorname{hocolim}_{\square_k} \mathbb{F}_k\left(C \bigsqcup A\right) \simeq \mathbb{F}_k(M) = \mathbb{F}_k\left(\operatorname{hocolim}\left(C \overset{i^-}{\underset{i^+}{\rightrightarrows}} A \right)\right).$$

Proof. In order to prove this theorem, we make an induction on k. If $k = 1$, the result is trivial. Now, assume that the geometrical realization of $X^{k-1}_\bullet$ has the same homotopy type as $\mathbb{F}_{k-1}(M)$. In order to apply Proposition 1.12, observe that $\mathcal{C}^k_p = \left(\mathcal{C}^{k-1}_{p-1} + \{k\}\right) \bigsqcup \mathcal{C}^{k-1}_p$. Hence, we have the equality:

$$\mathcal{C}^k_p \times \mathbb{F}_p(C) \times \mathbb{F}_{k-p}(A) = \mathcal{C}^{k-1}_{p-1} \times \mathbb{F}_p(C) \times \mathbb{F}_{k-p}(A) \bigsqcup \mathcal{C}^{k-1}_p \times \mathbb{F}_p(C) \times \mathbb{F}_{k-p}(A).$$

Consider the following diagram:

$$\begin{tikzcd}
\mathcal{C}^{k-1}_0 \times \mathbb{F}_1(C) \times \mathbb{F}_{k-1}(A) \arrow[r, shift left, "d_1^-"] \arrow[r, shift right, "d_1^+"'] & \mathcal{C}^{k-1}_0 \times \mathbb{F}_k(A) \\
\mathcal{C}^{k-1}_1 \times \mathbb{F}_2(C) \times \mathbb{F}_{k-2}(A) \arrow[u, shift left, "d_1^+"] \arrow[u, shift right, "d_1^-"'] \arrow[r, shift left, "d_2^-"] \arrow[r, shift right, "d_2^+"'] & \mathcal{C}^{k-1}_1 \times \mathbb{F}_1(C) \times \mathbb{F}_{k-1}(A) \arrow[u, shift left, "d_1^+"] \arrow[u, shift right, "d_1^-"'] \\
\mathcal{C}^{k-1}_2 \times \mathbb{F}_3(C) \times \mathbb{F}_{k-3}(A) \arrow[u, "d_2^+ \cdots d_1^-"] \arrow[r, shift left, "d_3^-"] \arrow[r, shift right, "d_3^+"'] & \mathcal{C}^{k-1}_2 \times \mathbb{F}_2(C) \times \mathbb{F}_{k-2}(A) \arrow[u, "d_2^+ \cdots d_1^-"] \\
\vdots \arrow[u, "d_3^+ \cdots d_1^-"] \arrow[r, dotted] & \vdots \arrow[u, "d_3^+ \cdots d_1^-"] \\
\mathcal{C}^{k-1}_{p-2} \times \mathbb{F}_{p-1}(C) \times \mathbb{F}_{k-p+1}(A) \arrow[u, "d_{p-2}^+ \cdots d_1^-"] \arrow[r, shift left, "d_{p-1}^-"] \arrow[r, shift right, "d_{p-1}^+"'] & \mathcal{C}^{k-1}_{p-2} \times \mathbb{F}_{p-2}(C) \times \mathbb{F}_{k-p+2}(A) \arrow[u, "d_{p-2}^+ \cdots d_1^-"] \\
\mathcal{C}^{k-1}_{p-1} \times \mathbb{F}_p(C) \times \mathbb{F}_{k-p}(A) \arrow[u, "d_{p-1}^+ \cdots d_1^-"] \arrow[r, shift left, "d_p^-"] \arrow[r, shift right, "d_p^+"'] & \mathcal{C}^{k-1}_{p-1} \times \mathbb{F}_{p-1}(C) \times \mathbb{F}_{k-p+1}(A) \arrow[u, "d_{p-1}^+ \cdots d_1^-"] \\
\vdots \arrow[u, "d_p^+ \cdots d_1^-"] \arrow[r, dotted] & \vdots \arrow[u, "d_p^+ \cdots d_1^-"] \\
\mathcal{C}^{k-1}_{k-2} \times \mathbb{F}_{k-1}(C) \times \mathbb{F}_1(A) \arrow[u, "d_{k-2}^+ \cdots d_1^-"] \arrow[r, shift left, "d_{k-1}^-"] \arrow[r, shift right, "d_{k-1}^+"'] & \mathcal{C}^{k-1}_{k-2} \times \mathbb{F}_{k-2}(C) \times \mathbb{F}_2(A) \arrow[u, "d_{k-2}^+ \cdots d_1^-"] \\
\mathbb{F}_k(C) \arrow[u, "d_{k-1}^+ \cdots d_1^-"] \arrow[r, shift left, "d_k^-"] \arrow[r, shift right, "d_k^+"'] & \mathcal{C}^{k-1}_{k-1} \times \mathbb{F}_{k-1}(C) \times \mathbb{F}_1(A) \arrow[u, "d_{k-1}^+ \cdots d_1^-"]
\end{tikzcd}$$

Label $C^k_\bullet$ the $h\square$-space on the left hand side of the diagram and $A^k_\bullet$ the $h\square$-space on the right hand side. Proposition 1.12 shows that there exists a $h\square$-space $|C^k_\bullet| \overset{d^-}{\underset{d^+}{\rightrightarrows}} |A^k_\bullet|$ and that its geometrical realization is the same as the one of $X^k_\bullet$. Furthermore, $|A^k_\bullet|$ is also augmented by the map $d_0 : \mathbb{F}_k(A) \hookrightarrow \mathbb{F}_k(M)$. This augmentation induces a map $\chi_A : |A^k_\bullet| \to \mathbb{F}_k(M)$ that fits in the following homotopy commutative diagram (‡):

$$\begin{tikzcd}
\mathbb{F}_k(A) \arrow[r, "d_0"] \arrow[d, hook] & \mathbb{F}_k(M) \\
{|A^k_\bullet|} \arrow[r, hook] \arrow[ur, "\chi_A"] & {|X^k_\bullet|} \arrow[u, "\chi"']
\end{tikzcd}$$

In order to study $A^k_\bullet$ and $C^k_\bullet$, define $\pi^M_k : \mathbb{F}_k(M) \longrightarrow M$ as the projection on the last coordinate of $\mathbb{F}_k(M)$, $\pi^A_k : \mathcal{C}^{k-1}_p \times \mathbb{F}_p(C) \times \mathbb{F}_{k-p}(A) \longrightarrow A$ as the projection on the last coordinate of $\mathbb{F}_{k-p}(A)$ and $\pi^C_k : \left(\mathcal{C}^{k-1}_{p-1} + \{k\}\right) \times \mathbb{F}_p(C) \times \mathbb{F}_{k-p}(A) \to C$ as the projection on the last coordinate of $\mathbb{F}_p(C)$. Observe that the $h\square$-space $A^k_\bullet$ is augmented by the map $\pi^A_k : \mathbb{F}_k(A) \to A$. Hence there is an induced map, still denoted π^A_k, from $|A^k_\bullet|$ to A. Also remark that $\iota \circ \pi^A_k = \pi^M_k \circ d_0$, and consequently the square

$$\begin{tikzcd}
{|A^k_\bullet|} \arrow[r, "\chi_A"] \arrow[d, "\pi^A_k"] & \mathbb{F}_k(M) \arrow[d, "\pi^M_k"] \\
A \arrow[r, "\iota"'] & M
\end{tikzcd}$$

is homotopy commutative. We claim that this square is a homotopy pullback. Indeed the map χ_A restricts to a homotopy equivalence between the homotopy fibers of the vertical maps. In order to show it, we know from V. Puppe [7] that the homotopy fiber of $\pi^A_k : |A^k_\bullet| \to A$ is the geometrical realization of the $h\square$-space obtained by restriction to the homotopy fibers. More precisely, the homotopy fiber of π^A_k over $a \in A$ is the geometrical realization of the $h\square$-space below:

$$\begin{tikzcd}
\mathcal{C}^{k-1}_0 \times \mathbb{F}_{k-1}(A \setminus r(a)) \\
\vdots \arrow[u, shift right, "d^+_1"] \arrow[u, shift left, "d^-_1"'] \\
\mathcal{C}^{k-1}_{p-1} \times \mathbb{F}_{p-1}(C) \times \mathbb{F}_{k-p}(A \setminus r(a)) \arrow[u, shift right=2, "d^+_{p-1}"] \arrow[u, shift left=2, "d^-_1"'] \\
\mathcal{C}^{k-1}_p \times \mathbb{F}_p(C) \times \mathbb{F}_{k-p-1}(A \setminus r(a)) \arrow[u, shift right=2, "d^+_p"] \arrow[u, shift left=2, "d^-_1"'] \\
\vdots \arrow[u, shift right=2, "d^+_{p+1}"] \arrow[u, shift left=2, "d^-_1"'] \\
\mathcal{C}^{k-1}_{k-2} \times \mathbb{F}_{k-2}(C) \times \mathbb{F}_1(A \setminus r(a)) \arrow[u, shift right=2, "d^+_{k-2}"] \arrow[u, shift left=2, "d^-_1"'] \\
\mathcal{C}^{k-1}_{k-1} \times \mathbb{F}_{k-1}(C) \arrow[u, shift right=2, "d^+_{k-1}"] \arrow[u, shift left=2, "d^-_1"']
\end{tikzcd}$$

By our induction hypothesis, the geometrical realization of this $h\Box$-space has the homotopy type of $\mathbb{F}_{k-1}(M \setminus r(a))$. Therefore, we get the annouced homotopy pullback. Using the same argument with $C^k_\bullet$, we show that the outer square of the following diagram is also a homotopy pullback:

$$\begin{tikzcd}
|C^k_\bullet| \arrow[r, "d^\epsilon"] \arrow[d, "\pi_k^C"] & |A^k_\bullet| \arrow[r, "\chi_A"] \arrow[d, "\pi_k^A"] & \mathbb{F}_k(M) \arrow[d, "\pi_k^M"] \\
C \arrow[r, "i^\epsilon"'] & A \arrow[r, "\iota"'] & M
\end{tikzcd}$$

Furthermore, we have the following equality of maps in $C^k_\bullet$:

$$d_1^\epsilon \circ \pi_k^C = r \circ i^\epsilon \circ \pi_k^C \simeq i^\epsilon \circ \pi_k^C = \pi_k^A \circ d_p^\epsilon : \mathcal{C}_{p-1}^{k-1} \times \mathbb{F}_p(C) \times \mathbb{F}_{k-p}(A) \to A.$$

Consequently, the left square is also a homotopy pullback. Let F be the common homotopy fiber of the three previous vertical arrows. According to Lemma 1.12, the geometrical realization of the $h\Box$-space $|C^k_\bullet| \underset{d^-}{\overset{d^+}{\rightrightarrows}} |A^k_\bullet|$ is the space $|X^k_\bullet|$ and the geometrical realization of $C \underset{i^-}{\overset{i^+}{\rightrightarrows}} A$ is M. Using [7], we know that the induced maps between those homotopy colimits $|X^k_\bullet| \to M$ has also F for homotopy fiber. Moreover, since the diagram (‡) is homotopy commutative, the square

$$\begin{tikzcd}
|X^k_\bullet| \arrow[r, "\chi"] \arrow[d, "\pi_k"] & \mathbb{F}_k(M) \arrow[d, "\pi_k"] \\
\left| C \underset{i^+}{\overset{i^-}{\rightrightarrows}} A \right| \arrow[r, "\simeq"] & M
\end{tikzcd}$$

is a homotopy pullback. Since the bottom map is a homotopy equivalence, the top map $\chi : |X^k_\bullet| \to \mathbb{F}_k(M)$ is also a homotopy equivalence. $\Box$

3 Application to braid groups

In this section, we explain the method to compute the pure braid groups of M and detail the case of the Möbius band.

3.1 Artin braid groups

The braid groups have been introduced by E. Artin [1]. Here we adopt the point of view where braid groups are defined in terms of fundamental group of a configuration space as given in [4] or, more generally, in [10]. It is known that $\mathbb{F}_k(\mathbb{R}^2)$ is an Eilenberg-MacLane space $K(P_k, 1)$ where P_k is called the *pure braid group on k strands*. We extend this definition as follows.

3.1 Definition
Let M be a connected manifold of dimension bigger than 2. The pure braid group on k strands of M is the group

$$P_k(M) = \pi_1(\mathbb{F}_k(M)).$$

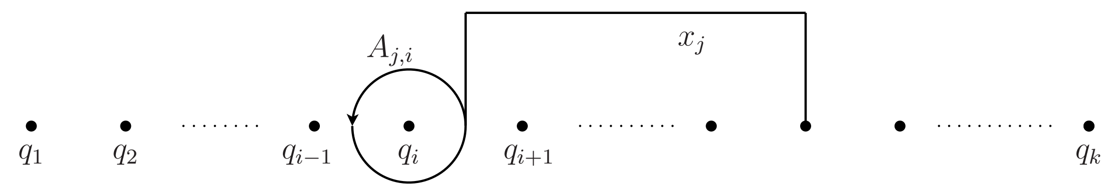
In this section, we give a description of the group P_k.

Let $\lceil 1,k \rfloor$ be the set of integers between 1 and k. Let $q_1 < q_2 < \cdots < q_k$ be some fixed real numbers. The image of q_* by the the natural inclusion $\mathbb{R} \cong \mathbb{R} \times \{0\} \subset \mathbb{R}^2$ is still denoted q_*. *The configuration* $Q_k = (q_1, \ldots, q_k)$ *is the base point of the space* $\mathbb{F}_k(\mathbb{R}^2)$. Finally, let $\epsilon = \inf_{i \neq j} |q_j - q_i|/2$ and $\vec{\epsilon} = (\epsilon, 0) \in \mathbb{R}^2$.

For $(i,j) \in \lceil 1,k \rfloor^2$, we define a path $t_{j,i} = (\tau_1, \ldots, \tau_k) : [0,1] \to \mathbb{F}_k(\mathbb{R}^2)$ joining the configuration $(q_1, \ldots, q_k)$ to the configuration $(q_1, \ldots, q_{j-1}, q_i + \vec{\epsilon}, q_{j+1}, \ldots, q_k)$ such that $\tau_r(t) = q_r$ if $r \neq j$ and $\tau_j(t) \in \mathbb{R} \times \mathbb{R}^+$. Define the loop $\alpha_{j,i} : S^1 \to \mathbb{F}_k(\mathbb{R}^2)$ by $\alpha_{j,i}(\xi) = (q_1, \ldots, q_{j-1}, q_i + \epsilon\xi, q_{j+1}, \ldots, q_k)$. The class $A_{j,i} \in \pi_1(\mathbb{F}_k(\mathbb{R}^2), Q_k)$ is defined as the homotopy class of the loop $t_{j,i} * \alpha_{j,i} * t_{j,i}^{-1}$ where $*$ denote the composition of paths. Observe that the homotopy class of $A_{j,i}$ is independent of the choice of $t_{j,i}$ and ϵ.

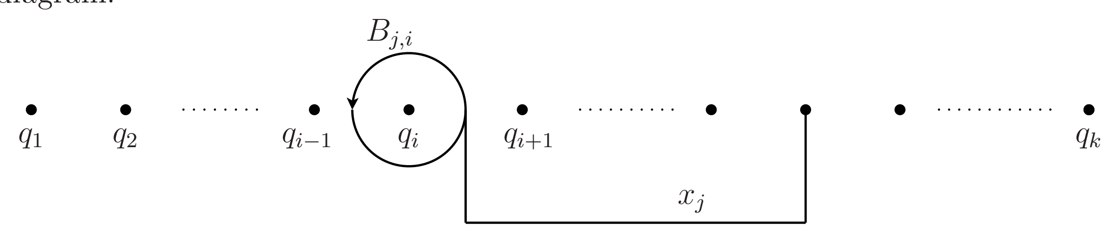
Whenever no confusion is possible, we denote in the same way the loops and their homotopy classes and we omit to write down the composition operation $*$.

The element $A_{j,i} = t_{j,i}\alpha_{j,i}t_{j,i}^{-1}$ is represented by the following diagram.



The r-th particle of the configuration is labeled x_r, except when it is fixed at the base point and is denoted q_r in that case.

Analogously, we define the class $B_{j,i}$ by requesting $\tau_j(t) \in \mathbb{R} \times \mathbb{R}^-$. It is represented by the next diagram.



The class $B_{j,i}$ is related to the classes $A_{*,*}$ by the relation

$$B_{j,i} = A_{j,j-1}^{-1} A_{j,j-2}^{-1} \cdots A_{j,i+1}^{-1} A_{j,i} A_{j,i+1} \cdots A_{j,j-2} A_{j,j-1}.$$

The following theorem gives a description of the group P_k.

3.2 Theorem
The group P_k admits the following presentation:
Generators : $A_{j,i}$ *with* $1 \leq i < j \leq k$.
Relations :

$$\begin{array}{lrl}
(1) & [A_{j,i}, A_{r,i}A_{r,j}] = 1 & \textit{if } \; 1 \leq i < j < r, \\
(2) & [A_{r,i}, A_{r,j}A_{j,i}] = 1 & \textit{if } \; 1 \leq i < j < r, \\
(3) & [A_{s,r}, A_{j,i}] = 1 & \textit{if } \; 1 \leq i < j < r < s, \\
(4) & [A_{s,i}, A_{r,j}] = 1 & \textit{if } \; 1 \leq i < j < r < s, \\
(5) & [A_{s,j}, A_{r,j}^{-1}A_{r,i}A_{r,j}] = 1 & \textit{if } \; 1 \leq i < j < r < s, \\
(6) & [A_{s,j}, A_{s,r}A_{r,i}A_{s,r}^{-1}] = 1 & \textit{if } \; 1 \leq i < j < r < s,
\end{array}$$

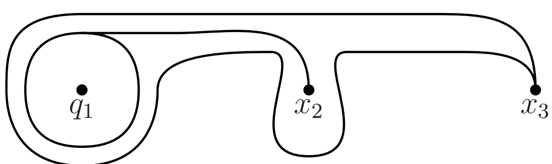
where the commutator $[A,B]$ *denotes the element* $A^{-1}B^{-1}AB$.

A complete proof of this theorem can be found in [4, 3]. It is also a consequence of the next lemma that we will use afterward.

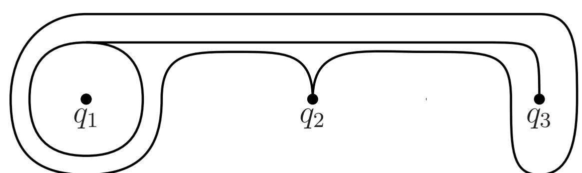
3.3 Lemma
Let M be a manifold and $Q_k = (q_1, \dots, q_k) \in \mathbb{F}_k(M)$ be a configuration in M.
Let U and V be two disjoint subsets of $\lceil 1, k \rfloor$.
Let $\tau = (\tau_1, \dots, \tau_k)$ be a path in $\mathbb{F}_k(M)$ starting at Q_k and such that τ_u is constant if $u \notin U$.
Let $\gamma = (\gamma_1, \dots, \gamma_k)$ be a loop in $\mathbb{F}_k(M)$ pointed at Q_k such that γ_v is constant if $v \notin V$.
Suppose that for all couple $(u, v) \in U \times V$ and all $(t_1, t_2) \in [0,1]^2$, we have $\tau_u(t_1) \neq \gamma_v(t_2)$.
Then the loop $\omega = (\omega_1, \dots, \omega_k) : [0,1] \to \mathbb{F}_k(M)$ defined by $\begin{cases} \omega_u = \tau_u * 1 * \tau_u^{-1} & \textit{if } u \in U \\ \omega_v = 1 * \gamma_v * 1 & \textit{if } v \in V \end{cases}$ is well defined and is homotopic to the path γ. In particular, if γ is a loop, we have $\omega = \tau\gamma\tau^{-1}$ and so $[\gamma, \tau] = 1 \in \pi_1(\mathbb{F}_k(M))$.

Proof. Let $s \in [0,1]$ and $\tau_{u,s}$ be the path in M defined by $\tau_{u,s}(t) = \tau_u(st)$. Define a loop $\Omega_s = (\omega_{1,s}, \dots, \omega_{k,s}) : [0,1] \to \mathbb{F}_k(M)$ by $\begin{cases} \omega_{u,s} = \tau_{u,s} * 1 * \tau_{u,s}^{-1} & \text{if } u \in U \\ \omega_{v,s} = 1 * \gamma_v * 1 & \text{if } v \in V \end{cases}$ where the path named 1 is the constant path at the right point. The path Ω_s is well defined because for every couple $(u, v) \in U \times V$ and all $t \in [0,1]$, we have $\tau_{u,s}(t) = \tau_u(st) \neq \gamma_v(t)$. Therefore, the loops $\omega = \Omega_1$ and $\Omega_0 = 1 * \gamma * 1 \simeq \gamma$ are in the same homotopy class. $\square$

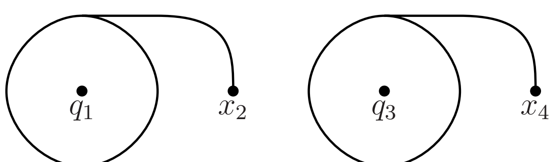
In the case of 3 or 4 particles, the proof of Theorem 3.2 is contained in the following diagrams.



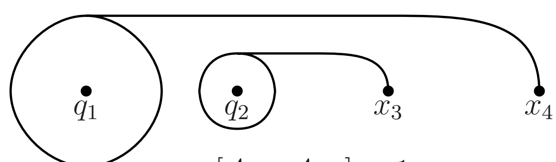
$[A_{2,1}, A_{3,1}A_{3,2}] = 1$



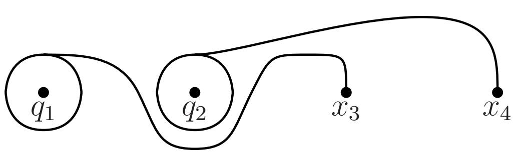
$[A_{3,1}, A_{3,2}A_{2,1}] = 1$



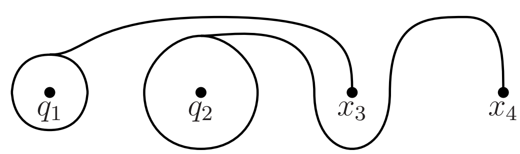
$[A_{4,3}, A_{2,1}] = 1$



$[A_{4,1}, A_{3,2}] = 1$



$[A_{4,2}, A_{3,2}^{-1}A_{3,1}A_{3,2}] = 1$



$[A_{3,1}, A_{4,3}A_{4,2}A_{4,3}^{-1}] = 1$

Figure: Yang-Baxter relations for 3 or 4 particles.

3.2 Practical determination of $P_k(M)$

As pointed out in Section 1, one does not need to know the rigorous structure of a $h\square$-space in order to calculate its fundamental group. To calculate the fundamental group of $|X_\bullet^k|$, which is $P_k(M)$, the truncated realization $|X_{\leq 2}^k|$ is enough. In fact, if $(X_2^k)^{(0)}$ denotes a 0-skeleton

for X_2^k, Proposition 1.11 asserts that the geometrical realization of the $h\square$-space

$$Y_\bullet :\ (X_2^k)^{(0)} \overset{d_1^-}{\underset{d_2^+}{\rightrightarrows\rightrightarrows}} X_1^k \overset{d_1^-}{\underset{d_1^+}{\rightrightarrows}} X_0^k$$

has the same fundamental group as $|X_\bullet^k|$ which is $P_k(M)$ by the Main Theorem. Here, we detail how this group can be described by generators and relations. Let $Y_\bullet$ be as above and denote by $Q_k \in \mathbb{F}_k(A) = Y_0$ the common base point of $|Y_{\leq 1}|$ and $|Y_{\leq 2}|$.

First step: The space $|Y_{\leq 1}|$ is the mapping torus

$$|Y_{\leq 1}| = \mathrm{hocolim}\left(\lceil 1,k \rfloor \times C \times \mathbb{F}_{k-1}(A) \overset{d_1^-}{\underset{d_1^+}{\rightrightarrows}} \mathbb{F}_k(A) \right).$$

In order to find the fundamental group of $|Y_{\leq 1}|$, we apply a Van Kampen like theorem inductively for each path-component of $Y_1 = \lceil 1,k \rfloor \times C \times \mathbb{F}_{k-1}(A)$. If a path component of $Y_1 = \lceil 1,k \rfloor \times C \times \mathbb{F}_{k-1}(A)$ is sent to two different path-components of $Y_0 = \mathbb{F}_k(A)$ by the maps d_1^- and d_1^+, we can use the usual Van Kampen theorem (see [5, Theorem 1.20]). In the other case, we use the following variation which can be easily proved.

3.4 Proposition (Van Kampen Theorem for a mapping torus)
Let f^- and f^+ be two maps between two arcwise connected spaces Z and Y. Let X be the homotopy colimit of the diagram

$$Z \overset{f^-}{\underset{f^+}{\rightrightarrows}} Y$$

i.e.

$$X = \frac{Y \bigsqcup Z \times [-1,1]}{f^+(z) \sim (z,+1), f^-(z) \sim (z,-1)}$$

Fix two points $y_0 \in Y$ and $z_0 \in Z$, a path y^+ in Y from y_0 to $f^+(z_0)$, a path y^- in Y from y_0 to $f^-(z_0)$, and z the path in X with support $\{z_0\} \times [-1,1]$. Then, there exists an isomorphism

$$\pi_1(X,y_0) \approx \frac{\pi_1(Y,y_0) \star < \rho >}{y^+ f^+(\omega)(y^+)^{-1} \sim \rho^{-1} y^- f^-(\omega)(y^-)^{-1}\rho \ \textit{for}\ \omega \in \pi_1(Z,z_0)}$$

where $\star$ denotes the free product of groups and $\rho \in \pi_1(X,y_0)$ is the loop $y^- z (y^+)^{-1}$.

Second step: In order to obtain $|Y_{\leq 2}|$, we glue a 2-cell on top of $|Y_{\leq 1}|$ for each point $x \in Y_2 = \left(\mathcal{C}_2^k \times \mathbb{F}_2(C) \times \mathbb{F}_{k-2}(A)\right)^{(0)}$. This 2-cell is attached along the loop $\Phi_2(x,\bullet) : S^1 \to |Y_{\leq 1}|$ described in 1.10. Also, for each point $x \in Y_2 = \left(\mathcal{C}_2^k \times \mathbb{F}_2(C) \times \mathbb{F}_{k-2}(A)\right)^{(0)}$, choose α_x a path in $|X_{\leq 1}|$ from $\Phi_2(x,1)$ to Q_k. Finally, as it is well known (see [5, Proposition 1.26]), the group $P_k(M) = \pi_1(|X_\bullet^k|, Q_k) = \pi_1(|Y_\bullet|, Q_k)$ admits for presentation

$$\pi_1(|X_{\leq 1}|, Q_k) \big/ < \alpha_x \Phi_2(x,\bullet) \alpha_x^{-1} \ \big|\ x \in Y_2 > .$$

3.3 Braids on the Möbius band $\mathcal{M}$

The Möbius band $\mathcal{M}$ can be represented as an embedding torus with C the interval $[-1,1]$ and A the square $[-1,1]\times[-1,1]$. The maps i^ϵ are defined by $i^\epsilon(x)=(\epsilon x,\epsilon)$.

We apply the method described in the previous section to $\mathcal{M}$. Observe that it is sufficient to describe the group $P_k(\mathcal{M})$ to know the homotopy type of $\mathbb{F}_k(\mathcal{M})$ since this space is a $K(P_k(\mathcal{M}),1)$. Indeed, it follows from the long exact sequence of the fibration

$$\bigvee_1^k S^1 \simeq \mathcal{M}\setminus\{q_1,\dots,q_{k-1}\}\to\mathbb{F}_k(\mathcal{M})\to\mathbb{F}_{k-1}(\mathcal{M})$$

and a trivial induction.

3.5 Theorem
The group $P_k(\mathcal{M})$ admits the following presentation:

- *Generators: ρ_i for $1\le i\le k$.*
- *Relations:*

$$\begin{array}{ll}
[\rho_i^{-1},\rho_j^{-1}]=[\rho_{j-1},\rho_j]\cdots[\rho_{i+1},\rho_j][\rho_j,\rho_i][\rho_j,\rho_{i+1}]\cdots[\rho_j,\rho_{j-1}] & \text{if } i<j,\\
[[\rho_i,\rho_j],\rho_r]=1 & \text{if } i<j<r,\\
[[\rho_i,\rho_r^{-1}],\rho_j]=1 & \text{if } i<j<r,\\
[[\rho_j,\rho_i],[\rho_r,\rho_i][\rho_r,\rho_j]]=1 & \text{if } i<j<r,\\
[[\rho_r,\rho_i],[\rho_r,\rho_j][\rho_j,\rho_i]]=1 & \text{if } i<j<r,\\
[[\rho_s,\rho_j],[\rho_j,\rho_r][\rho_r,\rho_i][\rho_r,\rho_j]]=1 & \text{if } i<j<r<s,\\
[[\rho_s,\rho_j],[\rho_s,\rho_r][\rho_r,\rho_i][\rho_r,\rho_s]]=1 & \text{if } i<j<r<s.
\end{array}$$

Moreover, the image of the generator $A_{j,i}\in P_k(A)$, $j>i$, by the map induced by the natural inclusion $A\hookrightarrow\mathcal{M}$ is $[\rho_j,\rho_i]$.

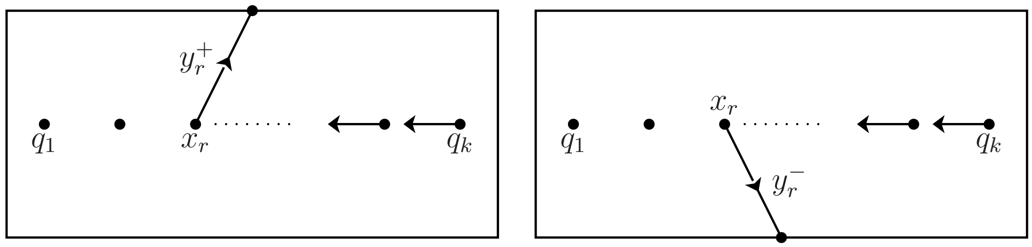
Proof. The group $P_k(\mathcal{M})$ is the fundamental group of the geometrical realization of the following $h\square_2$-space.

$$\left(\mathcal{C}_2^k\times\mathbb{F}_2(C)\times\mathbb{F}_{k-2}(A)\right)^{(0)} \rightrightarrows\!\!\rightrightarrows \mathcal{C}_1^k\times C\times(\mathbb{F}_{k-1}(A)) \rightrightarrows \mathbb{F}_k(A).$$

Let $s=(0,-1)\in\mathbb{F}_2(C)$ and $t=(0,1)\in\mathbb{F}_2(C)$. Up to an homotopy equivalence, the preceding diagram restricts to the following one named $Y_\bullet$.

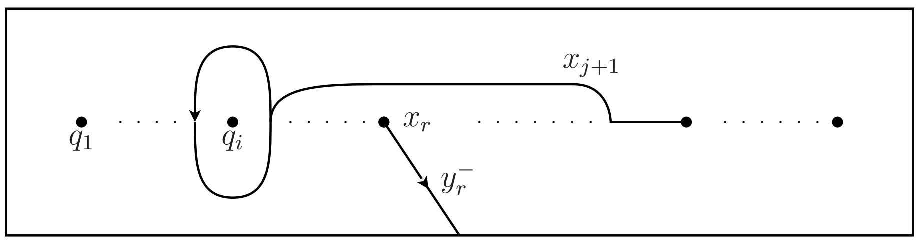
$$\mathcal{C}_2^k\times\{s,t\}\times\{(q_1,\dots,q_{k-2})\} \rightrightarrows\!\!\rightrightarrows \mathcal{C}_1^k\times\{0\}\times\mathbb{F}_{k-1}(A) \rightrightarrows \mathbb{F}_k(A)\,.$$

First step: We want to apply the Van Kampen Theorem 3.4 to obtain the fundamental group of $|Y_{\le 1}|$. So we fix $r\in\lceil 1,k\rfloor$ and consider the mapping torus $\{r\}\times\{0\}\times\mathbb{F}_{k-1}(A)\rightrightarrows\mathbb{F}_k(A)$. Define some paths in $\mathbb{F}_k(A)$, denoted by y_r^ϵ, moving the base point $Q_k=(q_1,\dots,q_{r-1},q_r,q_{r+1},\dots,q_k)$ to the configuration $(q_1,\dots,q_{r-1},i^\epsilon(0),q_r,\dots,q_{k-1})$ at constant speed along a segment like below.



The image of $A_{j,i} \in \pi_1(\{r\} \times \mathbb{F}_{k-1}(A)) \equiv P_{k-1}$, $1 \leq i < j \leq k$, in $\pi_1(\mathbb{F}_k(A), Q_k)$ by the map $y_r^- * d_1^-(\bullet) * (y_r^-)^{-1}$ is: $\begin{cases} A_{j,i} & \text{if} \quad i < j < r, \\ A_{j+1,i+1} & \text{if} \quad r \leq i < j, \\ A_{j+1,i} & \text{if} \quad i < r \leq j. \end{cases}$

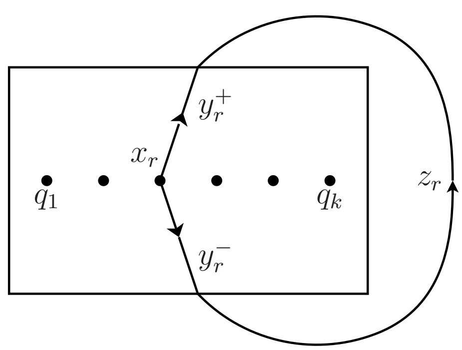
We prove the last line just above, the two others being similar. The loop $y_r^- * d_1^-(A_{j,i}) * (y_r^-)^{-1}$ is given by the following picture



and Lemma 3.3 shows that this loop is homotopic to $A_{j+1,i} \in P_k$. With a slight adaptation of the proof, we see that the image of $A_{j,i} \in \pi_1(\{r\} \times \mathbb{F}_{k-1}(A)) \equiv P_{k-1}$ in $\pi_1(\mathbb{F}_k(A), Q_k)$ by the map $y_r^+ * d_1^+(\bullet) * (y_r^+)^{-1}$ is: $\begin{cases} A_{j,i} & \text{if} \quad i < j < r, \\ A_{j+1,i+1} & \text{if} \quad r \leq i < j, \\ A_{j+1,r}^{-1} A_{j+1,i} A_{j+1,r} & \text{if} \quad i < r \leq j. \end{cases}$

Let z_r be the path in $|Y_{\leq 1}|$ with support the segment

$$r \times \{0\} \times (q_1, \ldots, q_{k-1}) \times [-1, 1] \subset \mathcal{C}_1^k \times C \times (\mathbb{F}_{k-1}(A)) \times [-1, 1] \subset |Y_{\leq 1}|.$$



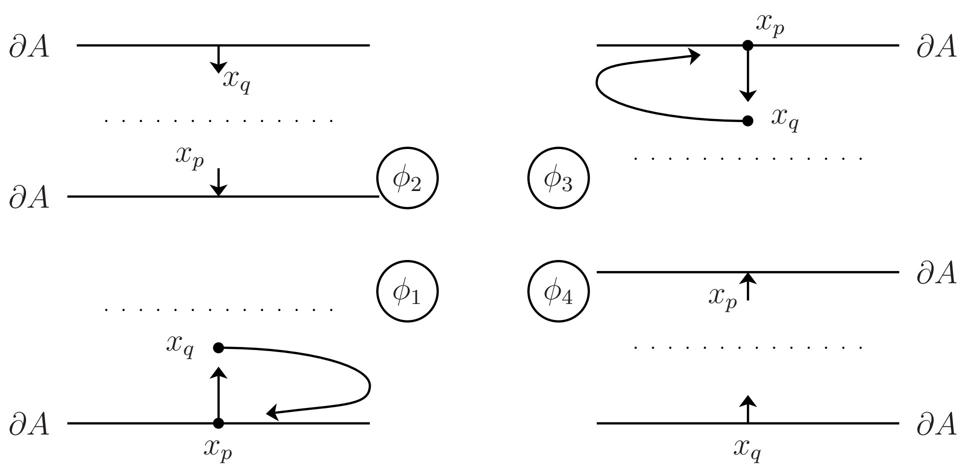
We finally define the loop $\rho_r = y_r^- z_r (y_r^+)^{-1} \in \pi_1(|Y_{\leq 1}|)$. Proposition 3.4 asserts that the space $|Y_{\leq 1}|$ admits the following presentation:

$$\mathcal{F}(\rho_1, \ldots, \rho_k) \star P_k / \mathcal{R}_1$$

where the relations $\mathcal{R}_1$ are given by:

$$\begin{aligned} &[A_{j,i}, \rho_r] = 1 && \text{if } i < j < r \text{ or } r < i < j, \\ &A_{j,r}^{-1} A_{j,i} A_{j,r} \rho_r^{-1} A_{j,i}^{-1} \rho_r = 1 && \text{if } i < r < j. \end{aligned}$$

Second step: The space $|Y_{\leq 2}|$ is obtained by gluing a 2-cell for each element of $\mathcal{C}_2^k \times \{s,t\} \equiv \mathcal{C}_2^k \times \{s,t\} \times \{(q_1, \dots, q_{k-2})\}$ to the space $|Y_{\leq 1}|$. Fix $(p,q) \in \mathcal{C}_2^k$. We carry out the details for the 2-cell associated to the point $((p,q),s)$.
First, we recall the paths ϕ_1, ϕ_2, ϕ_3 and ϕ_4 defined in Example 1.10. They are briefly schematized in the following way:



For the sake of clarity, the paths ϕ_1 and ϕ_3 are only pictured between the times 0 and $\frac{3}{4}$. Also, the points x_i for $i \notin \{p,q\}$, which stay motionless, are represented by a dotted line. Recall that the 2-cell is attached to the space $|Y_{\leq 1}|$ along the loop

$$\Phi_2(((p,q),s),\bullet) = \phi_1 D_1^- \phi_2 D_2^+ \phi_3 D_1^+ \phi_4 D_2^-$$

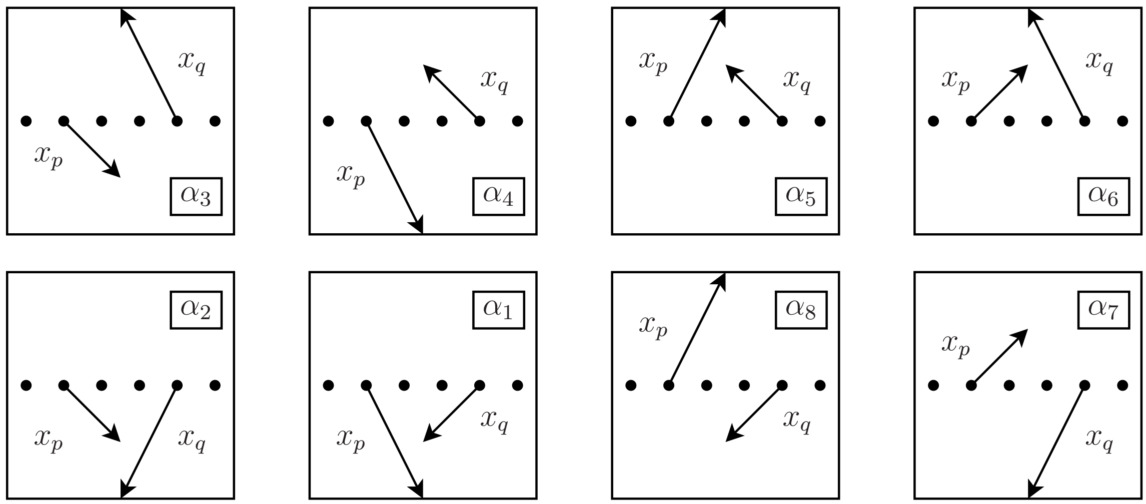
where the paths D_*^ϵ are defined like in Example 1.10. They are represented in the following way:

$$\begin{aligned} D_2^+ &= d_2^+((p,q),s) \times id, & D_1^+ &= d_1^+((p,q),s) \times (-id), \\ D_1^- &= d_1^-((p,q),s) \times id, & D_2^- &= d_2^-((p,q),s) \times (-id). \end{aligned}$$

A convenient way to study the homotopy class of this 2-cell attachment is to split the composition of path into a composition of well pointed loops. We define eight paths $\alpha_1, \dots, \alpha_8$ in $\mathbb{F}_k(A) = Y_0$ with origin the base point $(q_1, \dots, q_k)$ such that the following loop, named ω, is well defined:

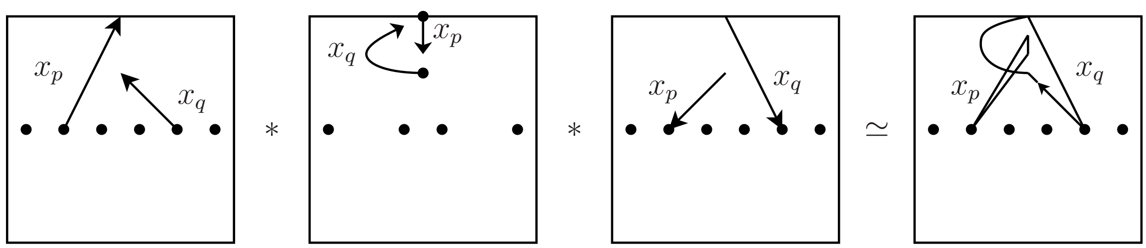
$$(\alpha_1 \phi_1 \alpha_2^{-1})(\alpha_2 D_1^- \alpha_3^{-1})(\alpha_3 \phi_2 \alpha_4^{-1})(\alpha_4 D_2^+ \alpha_5^{-1})(\alpha_5 \phi_3 \alpha_6^{-1})(\alpha_6 D_1^+ \alpha_7^{-1})(\alpha_7 \phi_4 \alpha_8^{-1})(\alpha_8 D_2^- \alpha_1^{-1})$$

In the rest of the proof, the paths α_* consist in moving the points of the configuration linearly and with constant speed as in the following pictures.

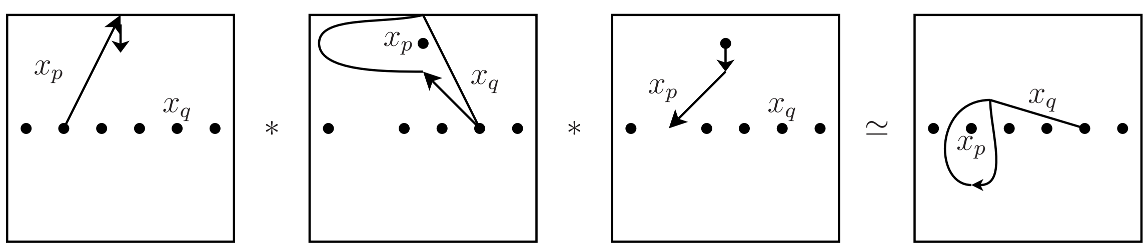


The 8 paths used for splitting ω

The final step for the determination of the homotopy class ω consists in drawing the various loops arising from the previous splitting. For example, consider the loop $\alpha_5\phi_3\alpha_6^{-1}$:

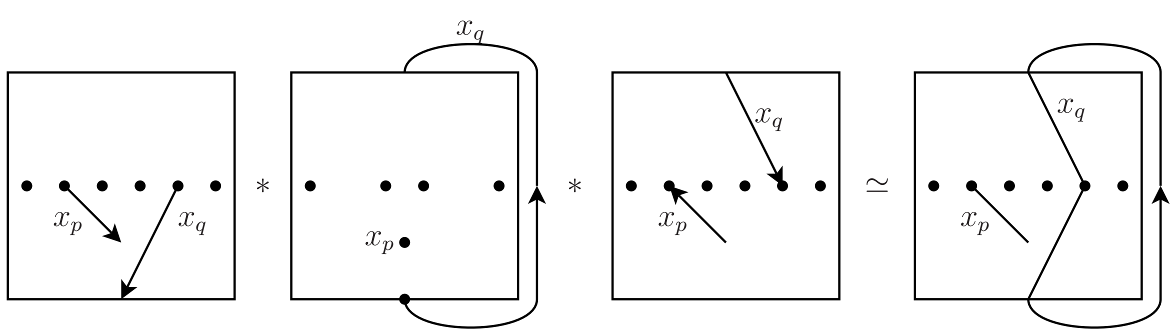


To find its homotopy class, observe that α_5 and α_6^{-1} can be re-parametrized such that the p-th point is moved before the q-th one. Hence, $\alpha_5\phi_3\alpha_6^{-1} \simeq A_{q,p}^{-1}$ as pictured below.



A direct application of Lemma 3.3 shows that the three loops $\alpha_1\phi_1\alpha_2^{-1}$, $\alpha_3\phi_2\alpha_4^{-1}$ and $\alpha_7\phi_4\alpha_8^{-1}$ are all nullhomotopic.

Also the maps d_1^- sends $((p,q),s)$ in $\{q\} \times \mathbb{F}_{k-1}(A)$. The loop $\alpha_2 D_1^- \alpha_3^{-1}$ is pictured below.



Observe that up to the position of the p-th particle, D_1^- is the path z_q. Hence, using the lemma 3.3, we come to the conclusion that $\alpha_2 D_1^- \alpha_3^{-1} \simeq \rho_q$. The cases $(\alpha_4 D_2^+ \alpha_5^{-1}) \simeq \rho_p$, $(\alpha_6 D_1^+ \alpha_7^{-1}) \simeq \rho_q^{-1}$ and $(\alpha_8 D_2^- \alpha_1^{-1}) \simeq \rho_p^{-1}$ are obtained analogously. Finally, the homotopy class of the loop ω is $\rho_q \rho_p A_{q,p}^{-1} \rho_q^{-1} \rho_p^{-1}$.

In a similar way, the 2-cell associated to the element $((p,q),t)$, once split and pointed at Q_k, is attached along the loop $B_{q,p} * \rho_q * 1 * \rho_p * 1 * \rho_q^{-1} * 1 * \rho_p^{-1} \simeq B_{q,p}\rho_q\rho_p\rho_q^{-1}\rho_p^{-1}$.

As proved in Section 3.2, the group $P_k(\mathcal{M}) = \pi_1(|Y_{\leq 2}|, Q_k)$ admits the presentation:

$$\mathcal{F}(\rho_1, \ldots, \rho_k) \star P_k / \mathcal{R}_2$$

where the relations $\mathcal{R}_2$ are given by:

$$\begin{array}{lll}
(a) & [A_{j,i}, \rho_r] = 1 & \text{if } i < j < r \text{ or } r < i < j, \\
(b) & A_{j,r}^{-1} A_{j,i} A_{j,r} \rho_r^{-1} A_{j,i}^{-1} \rho_r = 1 & \text{if } i < r < j, \\
(c) & B_{q,p} = [\rho_p^{-1}, \rho_q^{-1}] & \text{if } p < q, \\
 & A_{q,p} = [\rho_q, \rho_p] & \text{if } p < q.
\end{array}$$

The final step of the proof consists of a simplification of the presentation. For that, we first replace $A_{q,p}$ with $[\rho_q, \rho_p]$ in the previous relations.

(a) If $i < j < r$ or $r < i < j$, then $[A_{j,i}, \rho_r] = [[\rho_j, \rho_i], \rho_r] = 1$.

(b) If $i < r < j$, then $A_{j,r}^{-1} A_{j,i} A_{j,r} \rho_r^{-1} A_{j,i}^{-1} \rho_r = \rho_r^{-1}\rho_j^{-1}(\rho_r\rho_i^{-1}\rho_j\rho_i\rho_j^{-1}\rho_r^{-1}\rho_j\rho_i^{-1}\rho_j^{-1}\rho_i)\rho_j\rho_r = 1$
which implies $\quad \rho_r\rho_i^{-1}\rho_j\rho_i\rho_j^{-1}\rho_r^{-1}\rho_j\rho_i^{-1}\rho_j^{-1}\rho_i = 1 \quad$ i.e. $\quad [\rho_r, [\rho_i, \rho_j^{-1}]] = 1$.

(c) The relation $B_{j,i} = A_{j,j-1}^{-1} A_{j,j-2}^{-1} \cdots A_{j,i+1}^{-1} A_{j,i} A_{j,i+1} \cdots A_{j,j-2} A_{j,j-1}$ in P_k implies

$$[\rho_i^{-1}, \rho_j^{-1}] = [\rho_{j-1}, \rho_j][\rho_{j-2}, \rho_j] \cdots [\rho_{i+1}, \rho_j][\rho_j, \rho_i][\rho_j, \rho_{i+1}] \cdots [\rho_j, \rho_{j-2}][\rho_j, \rho_{j-1}].$$

Observe that, as a consequence of relation (a), we have $[[\rho_j, \rho_i], [\rho_s, \rho_r]] = [A_{j,i}, A_{s,r}] = 1$ for $i < j < r < s$ or $r < i < j < s$. In fact, these are two of the Yang-Baxter relations given in 3.2 ((3) and (4)). Finally, the remaining Yang-Baxter relations translate into the following relations.

$$\begin{array}{rcll}
[A_{j,i}, A_{r,i}A_{r,j}] = 1 & \Rightarrow & [[\rho_j, \rho_i], [\rho_r, \rho_i][\rho_r, \rho_j]] = 1 & \text{if } i < j < r, \\
[A_{r,i}, A_{r,j}A_{j,i}] = 1 & \Rightarrow & [[\rho_r, \rho_i], [\rho_r, \rho_j][\rho_j, \rho_i]] = 1 & \text{if } i < j < r, \\
[A_{s,j}, A_{r,j}^{-1}A_{r,i}A_{r,j}] = 1 & \Rightarrow & [[\rho_s, \rho_j], [\rho_j, \rho_r][\rho_r, \rho_i][\rho_r, \rho_j]] = 1 & \text{if } i < j < r < s, \\
[A_{s,j}, A_{s,r}^{-1}A_{r,i}A_{s,r}] = 1 & \Rightarrow & [[\rho_s, \rho_j], [\rho_s, \rho_r][\rho_r, \rho_i][\rho_r, \rho_s]] = 1 & \text{if } i < j < r < s.
\end{array}$$ □

Of course, we can iterate the construction by attaching a dimension 2 disc D^2 along the boundary of the Möbius band in order to obtain the projective plane $\mathbb{R}P^2$. In the particular case $k = 2$, we recover a result from J. van Buskirk[9].

3.6 Corollary ([9])
The pure braid group on 2 strands of the projective plane $\mathbb{R}P^2$ is isomorphic to the group of the quaternions Q_8.

References

[1] E. Artin. Theory of braids. *Ann. of Math. (2)*, 48:101–126, 1947.

[2] E. Fadell and L. Neuwirth. Configuration spaces. *Math. Scand.*, 10:111–118, 1962.

[3] E. R. Fadell and S. Y. Husseini. *Geometry and topology of configuration spaces*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2001.

[4] R. Fox and L. Neuwirth. The braid groups. *Math. Scand.*, 10:119–126, 1962.

[5] A. Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[6] J. W. Milnor and J. D. Stasheff. *Characteristic classes*. Princeton University Press, Princeton, N. J., 1974. Annals of Mathematics Studies, No. 76.

[7] V. Puppe. A remark on "homotopy fibrations". *Manuscripta Math.*, 12:113–120, 1974.

[8] G. Segal. Categories and cohomology theories. *Topology*, 13:293–312, 1974.

[9] J. van Buskirk. Braid groups of compact 2-manifolds with elements of finite order. *Trans. Amer. Math. Soc.*, 122:81–97, 1966.

[10] V. V. Vershinin. Braid groups and loop spaces. *Uspekhi Mat. Nauk*, 54(2(326)):3–84, 1999.

[11] R. M. Vogt. Homotopy limits and colimits. *Math. Z.*, 134:11–52, 1973.

[12] R. M. Vogt. Commuting homotopy limits. *Math. Z.*, 153(1):59–82, 1977.

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